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Variational problems in materials science: thin structures and defect patterns

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Abstract

This Ph.D. thesis addresses various problems arising from materials science and tackles them with techniques of the Calculus of Variations. The common theme is the presence of an energy—or a sequence of them—describing some physical system.

The thesis is divided in two parts. In the first one, we address three different elasticity problems for lower dimensional bodies, and we employ Γ-convergence as main tool. First, we derive a hierarchy of plate models for a singularly perturbed elastic energy allowing for different phases. Precisely, we assume that the elastic energy is minimized on a finite number of copies of $\mathrm{SO}(3)$, a setting that is useful to describe solid-solid phase transition. The singular perturbation is taken in such a way that only one phase is present when the thickness of the plate h goes to zero.

Then, we discuss the stability of the Von Kármán model for plates under loads of order h^2. The main novelty here is that we do not clamp the boundary of the plate, that is thus free to rotate. We derive a new compatibility condition between the limit force and the Von Kármán model. If this compatibility condition is not in force, then the Von Kármán model ceases to be valid.

Lastly, we derive a hierarchy of models for ribbons, starting from an intermediate, two-dimensional, elastic energy. The ribbon is modelled as a strip and its thickness has the role of a parameter in the energy. We show that this choice is well-suited to describe the behaviour of a ribbon, and we further investigate some scalings that are still open when starting from the three-dimensional model.

In the second part, we discuss two problems motivated by the study of dislocations, defects responsible for plastic response in metals. We first analyse an anisotropic nonlocal energy of Riesz type with physical confinement, that under certain conditions describes the interactions between edge dislocations. Such an energy can also be seen as an anisotropic variant of classical capacitary functionals in potential theory. Under suitable assumptions, we prove existence and uniqueness of minimizers, and we explicitly characterize them.

Then, we change framework, and we consider a two-dimensional rectangular cross-section of a crystal whose vertical boundaries are rotated of opposite small angles α. We show that, in a suitable modelling setting, a vertical grain boundary emerges and its energy scaling in α is consistent with the one predicted in the engineering literature.

1 Introduction

Even before our undergraduate studies, we are taught that many phenomena we observe can be effectively described by means of a principle of minimal energy. An object falling due to the gravitation force or a spring being pulled by our hand follow the same underlying principle: they are approaching the state of lowest energy—the ground state. For this reason, and many others, it was clear since the very first development of Calculus that techniques useful to characterize and/or find extremal points of functionals were crucial to improve our comprehension of the world.

The Calculus of Variations offers many techniques to find, characterize, and/or qualitatively study extremal—or critical—points of an energy functional. Their flexibility led to great advancements in many areas of mathematics, from geometry to regularity theory.

In this thesis, we focus on problems arising from materials science, and we employ techniques from the Calculus of Variations to study them. The common theme, suffice to say, is the presence of some underlying energy. The thesis is divided in two parts: in Part I, we address various elasticity problems for lower dimensional bodies, such as plates and ribbons; here Γ-convergence is the main tool we employ. In Part II, we answer questions arising from plasticity, with very different approaches, specific to the problem we analyse. The present introduction is thus divided in two parts.

1.1 Introduction to part I

Understanding the elastic response of a slender body is of great relevance in engineering and manufacturing. For example, it is important to know what kind of load a thin object can bear, how the small thickness is related to it, how boundary conditions influence the elastic response, etc. An extensive literature is devoted to the elasticity theory of slender bodies. Without claiming to be exhaustive, we recall the monographs [Lov27; Cia97a; Cia97b; Ant05; Lew23].

The different nature of thin objects makes the mathematical analysis extremely challenging. Different techniques might be employed for flat and non-flat bodies, such as shells and plates, or for objects with different dimensionality, as beams, rods, and ribbons. Besides the intrinsic difference between the various types of slender bodies, one may work in very different elastic settings, ranging from isotropic linear elasticity, to anisotropic hyperelasticity, contributing even more to the complexity of the field.

In the mechanical literature, one can find a great variety of properly lower-dimensional models that aim to describe effectively the elastic behaviour of an almost lower-dimensional object. As an example, many two-dimensional models have been proposed for plates, such as the Kirchhoff–Love equations [Kir50; Lov27], the Von Kármán's equations [Föp07; Von07; Von10], the membrane theory, the Reissner–Mindlin theory [Rei44; Rei44; Min51], the hierarchic plate theory, etc. (see also [Cia97a, Section 1.9] and the references contained therein). The main advantage of these models is that they are simpler to treat mathematically and, even more importantly, easier to handle numerically. However, in practical applications, any thin object is properly three-dimensional. Thus, it is natural to view these lower-dimensional models as approximations of a three-dimensional elastic model when one or more dimensions are small enough.

These considerations lead us to a natural question: given a slender body with some boundary conditions and/or applied loads, which model should we use to approximate its elastic behaviour? In other words, can we mathematically justify the validity of these models, under appropriate conditions? Indeed, many of the lower-dimensional models mentioned above are derived by means of some a priori assumptions, either of mechanical or geometrical nature, that we would like to rigorously justify or to derive a posteriori.

A successful mathematical approach, which is the one we employ in this thesis, is based on Γ-convergence. This is a variational notion of convergence for sequences of functionals introduced by De Giorgi in [DF75] (see also [Dal93; Bra06] for a more modern treatment). The main property that makes Γ-convergence well suited to tackle our question is the following: once Γ-convergence is proved with respect to some topology T, if the sequence of functionals is equicoercive in T, then (quasi)-minimizers converge to minimizers of the Γ-limit, where the convergence is in the topology T. Roughly speaking, this means that ground states (or solutions) to the three-dimensional problem are good approximations, in the sense of T, of the ground states (or solutions) of the lower-dimensional problem.

It is clear that the choice of the topology is a critical issue. Indeed, we should pick one coarse enough to have equicoercivity, but finer enough to prove Γ-convergence. Chapters 3 to 5 have thus the same structure: firstly, we choose the topology that is naturally induced by the energy to ensure equicoercivity, then we move to the proof of Γ-convergence.

In this thesis we focus on two kinds of slender bodies: plates in Chapters 3 and 4 and ribbons in Chapter 5.

1.1.1 Rigorous derivation of plate models

In recent years, a vast literature has been devoted to the rigorous derivation of plate models by means of Γ-convergence. To better understand these results, let us introduce some notation. Consider the reference configuration of a hyperelastic thin plate $\Omega_h := S \times (-h/2, h/2)$, where the mid-plane S is a sufficiently regular subset of $\mathbb{R}^2$ and h is the small thickness. Given a deformation $w : \Omega_h \to \mathbb{R}^3$, its elastic energy takes the form

$$\int_{\Omega_h} \mathcal{W}(\nabla w)\, dx,$$

where $\mathcal{W}$ is the elastic energy density. Since a thin plate can easily undergo large rotations, the correct framework is that of nonlinear elasticity. We assume that Ω_h is not prestrained, that is, Id minimizes $\mathcal{W}$ and we suppose $\mathcal{W}$ to be frame indifferent, namely

$$\mathcal{W}(RM) = \mathcal{W}(M) \qquad \forall R \in \mathrm{SO}(3), \forall M \in \mathbb{R}^{3\times 3}. \tag{1.1}$$

In particular, $\mathcal{W}$ is minimized at SO(3). Property (1.1) has a simple physical interpretation: the energy is invariant under rigid changes of the reference frame. We may assume the body to be subject to some dead loads f_h acting on the bulk. Hence, the total energy has the form

$$\int_{\Omega_h} \mathcal{W}(\nabla w)\, dx - \int_{\Omega_h} f_h \cdot w\, dx.$$

Note that the forcing term involves the deformation and not the displacement, since this change has no effect from a minimization standpoint.

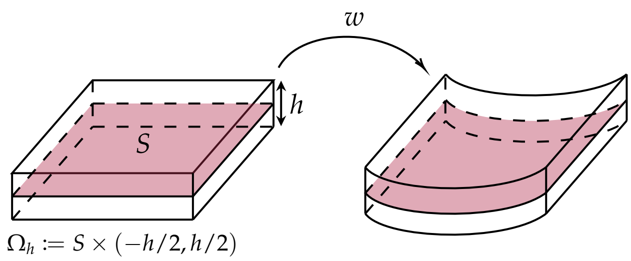

Figure 1.1: A plate and its deformation.

We also introduce the energy per unit volume, and write it in terms of a rescaled deformation as

$$I_h(y) := \frac{1}{h} \int_{\Omega_h} \mathcal{W}(\nabla w)\, dx = \int_{\Omega} \mathcal{W}(\nabla_h y)\, dx,$$

where $y(x_1, x_2, x_3) := w(x_1, x_2, hx_3)$, and ∇_h is the rescaled gradient (see Section 2.1 for its definition).

The first Γ-convergence result for plates in this setting is due to [LR95], where the authors study the Γ-limit of

$$\frac{1}{h} \int_{\Omega_h} \mathcal{W}(\nabla w)\, dx - \frac{1}{h} \int_{\Omega_h} f_h \cdot w\, dx = I_h(y) - \int_{\Omega} \tilde{f}_h \cdot y\, dx,$$

when the plate is clamped on a portion of the boundary and the load f_h is supposed to be of order 1. Here, $\tilde{f}_h(x_1, x_2, x_3) := f_h(x_1, x_2, hx_3)$. When $h \to 0$ they obtain the membrane theory: the limiting deformations are properly two-dimensional and their energy depends solely on the stretching they produce on S and does not account for bending. The Γ-convergence is obtained assuming some p-growth condition from above on the energy density $\mathcal{W}$. We mention also the further improvement in [AM08], where the same result is proved under the more physical assumption that $\mathcal{W}(M) = +\infty$ if $\det(M) \leq 0$.

As noted in [LR95, Theorem 10], compression requires no energy at the limit in the membrane theory. For example, if S is a rectangle, the minimum energy under uniaxial compressive boundary conditions scales like h^3 (see also [FJM02, Section 6]). Thus, one should look at the Γ-limit of

$$\frac{1}{h^3}\int_{\Omega_h} \mathcal{W}(\nabla w)\,dx = \frac{1}{h^2} I_h(y). \tag{1.2}$$

In contrast with the membrane regime, here the energy has a higher scaling in h. Assuming $\mathcal{W}$ to be minimized only at $\mathrm{SO}(3)$, this heuristically means that the deformation gradients should approach, in some suitable sense, the set of rotations. In order to compute the Γ-limit, quantitative information about this rate of convergence is crucial.

The needed breakthrough was obtained by Friesecke, James, and Müller in [FJM02], where they proved both a rigidity estimate, and the Γ-convergence of (1.2) to the Kirchhoff–Love energy under compatible boundary conditions and with loads f_h of order h^2 (the latter case being considered in the subsequent work [FJM06]).

The rigidity estimate gives a quantitative version of the well-known result by Liouville (see also [Res67] for a nonquantitative generalization of Liouville's theorem) stating that a map w satisfying the differential inclusion $\nabla w \in \mathrm{SO}(3)$ is a rigid rotation, i.e., $w(x) = Rx + c$ for some $R \in \mathrm{SO}(3)$, $c \in \mathbb{R}^3$. Precisely, given a domain $\Omega \subset \mathbb{R}^3$, there is a constant $C > 0$, depending on Ω, such that, for every $w \in W^{1,2}(\Omega;\mathbb{R}^3)$ there exists a rotation $R \in \mathrm{SO}(3)$ satisfying

$$\|\nabla w - R\|^2_{L^2} \leq C\|\operatorname{dist}(\nabla w, \mathrm{SO}(3))\|^2_{L^2}. \tag{1.3}$$

The estimate above can also be interpreted as a nonlinear version of Korn's inequality, since the tangent space to $\mathrm{SO}(3)$ at the identity is the space of skew-symmetric matrices $\mathbb{R}^{3\times 3}_{\mathrm{skew}}$.

For the thin domain Ω_h, it can be seen that the constant C in (1.3) behaves as h^{-2}. Thus, assuming the coercivity condition

$$\mathcal{W}(M) \geq C\operatorname{dist}^2(M; \mathrm{SO}(3)) \qquad \forall\, M \in \mathbb{R}^{3\times 3}, \tag{1.4}$$

and rewriting (1.3) in the scaled variables, we see that a bound on the rescaled energy (1.2) provides a uniform control in h on the distance of $\nabla_h y$ to some rotation.

Using this result, one can show that the limit deformations obtained in this case are isometric immersions of the mid-plane in $\mathbb{R}^3$. This is consistent with the a priori assumption of the Kirchhoff–Love theory saying that the mid-plane remains unstreched. The Γ-limit weights the second fundamental form of the isometric immersion, thus it penalizes bending.

It is natural to wonder what limiting behaviour emerges from a rescaling of I_h by a power $h^{-\alpha}$ with $0 < \alpha < 2$ and $\alpha > 2$.

For $0 < \alpha < 5/3$, the admissible limit deformations are the so-called short maps: immersions y of S in $\mathbb{R}^3$ such that $\nabla y^T \nabla y \leq \mathrm{Id}$ (see [CM07]). The energy is trivial on the set of admissible deformations. In this case, the theory is also called constrained membrane theory.

The Γ-convergence of $h^{-\alpha} I_h$, for $5/3 \leq \alpha < 2$ seems to be out of reach at the present time (see [GO97; Bel+02; JS01; CM07] for related results).

The case $\alpha > 2$ is treated in [FJM06]. As we may expect, also in this regime the rigidity estimate plays a crucial role. The authors prove that, up to a rotation and to a translation, deformations with bounded energy converge to the identity. Thus, the quantity of interest is now the deviation from the identity, namely, the displacement. The authors show that the x_3 average of the in-plane displacements u_h and of the out-of-plane displacements v_h have different scalings. Precisely, $u_h \sim \max\{h^{2\gamma-2}, h^\gamma\}$ and $v_h \sim h^{\gamma-1}$, where $\gamma := \alpha/2$. The regime $\alpha = 4$ is a threshold in the behaviour of u_h and, as a consequence, in the behaviour of the energy, too.

When $2 < \alpha < 4$, the limit u of the rescaled in-plane displacements and the limit v of the rescaled out-of-plane displacements satisfy the constraint

$$\nabla u^T + \nabla u + \nabla v \otimes \nabla v = 0, \tag{1.5}$$

and the energy is quadratic in $\nabla^2 v$. Constraint (1.5) has a geometric interpretation. Precisely, u and v satisfy a so-called matching isometry condition up to the second order, that is, the map

$$y_\varepsilon := \begin{pmatrix} x' \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ v \end{pmatrix} + \varepsilon^2 \begin{pmatrix} u \\ 0 \end{pmatrix}$$

satisfies $\nabla y_\varepsilon^T \nabla y_\varepsilon - \mathrm{Id} = O(\varepsilon^3)$. It can be proved that (1.5) is equivalent to the Monge–Ampère equation $\det(\nabla^2 v) = 0$ and that the latter ensures the existence of an exact isometry with v as third component. This theory has been named by the authors as the constrained Von Kármán's theory.

If $\alpha = 4$, they retrieve the classical Von Kármán's energy. Constraint (1.5) is relaxed, and the deviation from it appears as a stretching term in the energy, together with a quadratic term in $\nabla^2 v$, accounting for bending.

When $\alpha > 4$ they obtain the usual linear theory, with the energy being quadratic in both $\nabla^2 v$ and $\nabla u^T + \nabla u$.

We also mention the recent preprint [FGZ25], where the Reissner–Mindlin theory is rigorously derived.

Multi-well energies

All the Γ-convergence results we recalled for $\alpha \geq 2$ are obtained under the assumption that $\mathcal{W}$ is minimized exactly at the set of rotations. This is a fundamental hypothesis to apply the rigidity estimate (1.3). In Chapter 3 of this manuscript, we are interested in treating the case where $\mathcal{W}$ is minimized at a finite number of copies of SO(3), that we call wells. Precisely, we assume that $\mathcal{W}$ is minimized at the set

$$K := \bigcup_{i=1}^{l} \mathrm{SO}(3)U_i =: \bigcup_{i=1}^{l} K^i,$$

where $U_1, \ldots, U_l$ are symmetric and positive-definite matrices. This setting is relevant, for example, in the modelling of solid-solid phase transitions (see [BJ87]).

Assuming (1.4) with SO(3) replaced by K, the boundedness of the energy would grant us L^2-closeness of the deformation gradients to K. However, this is not enough to get compactness for sequences of deformations with bounded energy. Indeed, it is well-known that the rigidity estimate fails without further assumptions on the wells. For example, if two of the wells are rank-one connected, it is possible to construct continuous deformations with zero elastic energy, whose gradient oscillates between the two wells.

A rich literature has been devoted to the extension of the rigidity estimate to the multi-well setting under suitable separation properties for the wells. For two strongly incompatible wells, the rigidity estimate has been proved in [CM04] (and later with a different proof in [DS06]). A further generalization is given in [CC10], for a finite number of well-separated wells.

One possible way to overcome the loss of compactness is to singularly perturb the elastic energy by adding a higher order term of the form

$$\eta^p(h) \int_\Omega |\nabla_h^2 y|^p \, dx \, ,$$

where $\eta(h) \to 0$ as $h \to 0$ and $p > 1$ is a suitable exponent. This is a classical way of selecting preferred configurations, as for instance in the Van der Waals–Cahn–Hilliard theory. Indeed, this additional term introduces a competition in the minimization problem: the elastic energy favours deformations with gradient in K, while the perturbation penalizes transitions between the wells.

Various analyses have been carried out in this setting. In the membrane scaling $\alpha = 0$, a full description of the Γ-limit of the perturbed energy with $p = 2$ is obtained in [FFL06; DFL10]. The expression of the Γ-limit depends on the behaviour of $\eta(h)/h$. We also recall the work by Shu [Shu00], where an additional homogenization parameter is taken into account. Note that in the original work on membrane theory [LR95] no conditions on the minimization set $\mathcal{W}$ were required. In particular, no penalty term is needed to derive membrane theory in case of a multi-well energy. We also recall the work [FP04], where Γ-convergence in the membrane regime is computed in a different topology. Using the notion of Young's measure, the authors manage to preserve information on the fine oscillations between the wells.

In Chapter 3, we focus on the energy regimes $\alpha \geq 2$, and we do not assume any hypotheses of separation or connectedness of the wells. We follow ideas from [Ali+18], where the linearization of a multi-well elastic energy is studied. We show that the perturbation coefficient $\eta(h)$ can be chosen in such a way that, at the limit, deformation gradients are forced to fall into a single well and at the same time the perturbation term becomes negligible. Under these scaling assumptions on $\eta(h)$, we show that the L^2-norm of the distance of the deformation gradient from a specific well can be bounded by a suitable power of the perturbed elastic energy (see Section 3.2). In particular, once such a well has been identified, the usual rigidity estimate can be applied to deduce compactness.

The hierarchy of plate models that we derive is similar to the one discussed earlier. However, a key difference is the dependence of the limit models on the well around which we linearize.

For $\alpha = 2$, we retrieve the Kirchhoff's model (see Theorem 3.1.1). Sequences of deformations with $I_h \sim h^2$ converge to isometric immersions of the mid-plane into $\mathbb{R}^3$, for a flat metric depending on the well. The resulting model is similar in spirit to the one obtained in [BLS16], where the authors consider a prestrained model and the limit deformation realizes an isometric immersion of the mid-plane for a metric depending on the prestrain.

For $2 < \alpha < 4$, the Γ-limit is given by the constrained Von Kármán's model, where the Von Kármán's constraint now depends on the well and takes the form

$$\nabla u^T + \nabla u + |U_i^{-1} e_3|^2 \nabla v \otimes \nabla v = 0\,. \tag{1.6}$$

Here $SO(3)U_i$ is the well selected by the energy, and u and v are the limiting in-plane and out-of-plane displacements with respect to the reference deformation $x \mapsto U_i x$. As for the single-well case treated in [FJM06], (1.6) is a matching isometry condition up to the second order, for a well-dependent flat metric (see Section 3.1).

For $\alpha = 4$ and $\alpha > 4$, we retrieve the Von Kármán's and the linearized Von Kármán's model, respectively. All the Γ-convergence results for $\alpha > 2$ are stated in Theorem 3.1.3. The Von Kármán's model we obtain is similar to the one derived in [RLR17] in the prestrained case.

Concerning the proofs, once compactness is established, the liminf inequality can be obtained arguing as in [FJM02; FJM06]. Since the penalty term requires additional regularity, the truncation argument used in [FJM02; FJM06] for the construction of the recovery sequences cannot be applied and is replaced by suitable approximation results. In particular, for $2 \le \alpha < 4$ we need to assume some higher regularity for the mid-plane S.

The last part of Chapter 3 is devoted to the convergence of (quasi-)minimizers for the pure traction problem, under the action of some dead loads f_h. We assume f_h to be of order $h^{\gamma+1}$, where $\gamma = \alpha/2$. For $\alpha > 2$ we expect the limiting force term to depend only on the out-of-plane displacement (which is of order $h^{\gamma-1}$) and not on the in-plane displacement (which is of order $h^{2\gamma-2}$ for $2 \le \alpha \le 4$, h^γ for $\alpha > 4$). This is indeed the case. However, since the plate is not clamped, the presence of the force term may reduce the rotation invariance of the problem. To analyse this issue, we follow the approach used in [MM21], where the notion of optimal rotations is introduced. These are rotations preferred by the forces, that can be different from the rotations selected by the rigidity estimate, around which linearization takes place. We show how this concept can be adapted to the dimension reduction setting. In this framework, we deduce a minimization property for the limit of (quasi-)minimizing sequences. The forcing term we obtain at the limit is of the form

$$\int_\Omega f \cdot R U_j^{-1} e_3 v \, dx,$$

where R is an optimal rotation, U_j is the well selected by the energy and v is the limit of the rescaled out-of-plane displacements. In particular, the force acting on ve_3 is of the form $U_j^{-1}R^T f$. A precise statement of these results is given in Theorems 3.1.6 and 3.1.8.

The content of Chapter 4 can also be found in [Tol25a].

The pure traction problem for the Von Kármán's model

After its rigorous derivation, the Von Kármán's model has received great attention. Without attempting to be exhaustive, we recall some lines of research: derivation of viscoelastic Von Kármán's models for plates [FK20], homogenization of Von Kármán's plates models [Vel16; NV13], and analysis in the dynamic case of the Von Kármán's equations [AMM10; AMM11].

In [FJM06] the authors show that the derivation of the Von Kármán's model, corresponding to the energy regime $\alpha = 4$, is compatible with loads f_h of order h^3 in the normal direction to the plate. A natural question is whether forces acting in any direction can be included in this analysis. Since the in-plane displacements scale as h^2 the applied loads in the planar direction should be of order h^2 to be preserved at the limit. Howevere, such a choice is compatible not only with the Von Kármán's regime, but also with the Kirchhoff's and the constrained Von Kármán's regimes. This is due to the fact that in the last two cases, we have, respectively,

(i) in-plane displacements of order 1 and elastic energy of order h^2,

(ii) in-plane displacements of order $h^{\alpha-2}$ and elastic energy of order h^α for some $2 < \alpha < 4$.

Therefore, in both scenarios the work done by the forces and the elastic energy have the same scaling. Thus, a sequence of deformations y_h with total energy of order h^4 may have elastic energy that scales as h^α for any $2 \leq \alpha \leq 4$, leading to different limiting behaviours. In particular, if $\alpha < 4$, such a sequence has unbounded elastic energy in the Von Kármán's regime, resulting in a loss of compactness. This phenomenon can be interpreted as an instability of the Von Kármán's model under the presence of some load (see [LM09]).

As we already remarked, the situation is different when the applied forces are purely normal. Indeed, in this case, the h^3 scaling for forces is only compatible with the Von Kármán regime, where the normal displacement of order h. As a consequence, there is no ambiguity between the elastic energy regimes.

Planar forces have been considered in [LM09] using a clever exclusion principle. The authors noted that there is a critical load f that leads to the loss of validity of the Von Kármán's model. Under some additional assumptions, they also proved that beyond this critical load, the infimum of the total Von Kármán's energy is $-\infty$. However, to avoid the mix-up of planar and normal components of both forces and displacements due to rotation invariance, they assumed part of the boundary to be clamped.

In Chapter 4, we extend this analysis to the purely Neumann case. Since the body is free to rotate, one cannot distinguish between normal and planar components of the applied forces. Thus, we suppose to have a sequence of forces f_h that scale as h^2 in all directions. For simplicity, we further assume the sequence to be of the form $f_h = h^2 f$ for some given f.

The first question to understand is how the load affects the rotation invariance of the plate. In general, one cannot expect the body to prefer just one specific rotation, in contrast with the case of clamped boundary conditions. It turns out that the concept of optimal rotations, already recalled in Chapter 3, is exactly the one needed. The set $\mathcal{R}$ of such rotations is a submanifold of SO(3) that in our framework enjoys some additional properties which follow by the two-dimensional nature of the problem.

Secondly, we investigate how the stability conditions defined in [LM09] can be extended and how they relate to the rotational degree of freedom that the plate enjoys. We prove that one of the following alternatives holds (see Theorem 4.1.3 for a precise statement):

(i) either the load is strong enough to have a non-trivial minimizer of the Kirchhoff model (failure of the stability condition (S1)),

(ii) or the load is strong enough to have a non-trivial minimizer of the constrained Von Kármán's model (failure of the stability condition (S2)),

(iii) or the Von Kármán's model is valid.

This result is similar in spirit to [LM09, Theorem 4]. Moreover, in Theorem 4.1.4 we show that the stability condition (S1) implies condition (S2) as soon as the intensity of the load decreases. The above implication is analogous to [LM09, Theorem 6].

Compared to the analysis in [LM09], we observe a new phenomenon, which is one of the main novelties of this work: if for some optimal rotation R we have $R^T f \cdot e_3 \neq 0$, then the stability condition (S1) must fail and both the Von Kármán's model and its constrained version do not apply. More precisely, whenever $R^T f \cdot e_3 \neq 0$, every sequence of quasi-minimizers, whose total energy scales like h^4, has unbounded elastic energy in both the Von Kármán's and the constrained Von Kármán's regimes. The privileged role of e_3 is due to it being the direction along which the plate is thin. The precise statement is given in Theorem 4.1.2. One can interpret this result in the following way: it is possible to have a non-trivial minimizer of the Kirchhoff model either increasing the load (as already shown in [LM09]) or applying a force for which there is an optimal rotation R such that $R^T f \cdot e_3 \neq 0$.

Lastly, similarly to [LM09, Theorem 27], we prove that if (S2) holds and $R^T f \cdot e_3 = 0$ for every optimal rotation, the total Von Kármán's energy attains its infimum. Conversely, if (S2) fails, the Von Kármán's total energy is unbounded as soon as the load undergoes a slight increase. In other words, f is a critical load. These results are proved in Theorem 4.1.5.

The content of Chapter 4 can also be found in [Tol25b].

1.1.2 Rigorous derivation of models for ribbons

A ribbon is a slender body whose length is much larger than its width, which in turn is much larger than its thickness. Mathematically, it can be described by a set

$$\Omega_h := (0, L) \times (-h/2, h/2) \times (-\delta_h/2, \delta_h/2),$$

with $L > 0$ being its length, $h \ll L$ being its width, and $\delta_h \ll h$ its thickness.

A renewed interest in the rigorous derivation of one-dimensional models for ribbons via Γ-convergence has blossomed in recent years, motivated by the rich variety of behaviours that may emerge, depending on the behaviour of δ_h with respect to h. We recall also the setting $\delta_h \sim h$ for the description of rods, treated in [MM03; MM04; Sca06; Sca09].

Assuming the ribbon to be hyperelastic, the energy per unit volume takes the form

$$I_h(w) := \frac{1}{h\delta_h} \int_{\Omega_h} \mathcal{W}(\nabla w)\, dx = \int_{\Omega} \mathcal{W}(\nabla_{h,\delta_h} y)\, dx,$$

where $y(x_1, x_2, x_3) := w(x_1, hx_2, \delta_h x_3)$, $\Omega := (0, L) \times (-1/2, 1/2)^2$, and ∇_{h,δ_h} is the rescaled gradient.

Inspired by the derivation of plate theory in [FJM02; FJM06], the Γ-convergence of various rescalings of I_h has been studied in [FMP12; FMP13]. Assuming the energy I_h to scale as ε_h^2, the authors distinguish three regimes:

(i) the subcritical regime, for $\delta_h \ll \varepsilon_h$,

(ii) the critical regime, for $\delta_h \sim \varepsilon_h$,

(iii) the supercritical regime, for $\delta_h \gg \varepsilon_h$.

The subcritical regime corresponds roughly to the scaling $0 \leq \alpha < 2$ for plates. Indeed, if $\varepsilon_h \sim 1$ they retrieve the model already derived in [ABP91] for strings, that inspired the work [LR95] for plates. The limit energy depends only on stretching and does not account for bending. When instead $\varepsilon_h \to 0$, the limiting deformation are short maps defined on the mid-line, and the limiting energy is zero on the set of such deformations.

The critical regime corresponds to the Kirchhoff–Love theory for plates. However, the authors manage to compute the Γ-limit only for $h^2 \ll \delta_h$. In this case, they show that sequences of deformations with bounded energy identify a Frenet–Serrin frame d_1, d_2, d_3 describing an isometric immersion of the mid-line in $\mathbb{R}^3$. The frame satisfies the additional constraint

$$\partial_1 d_1 \cdot d_2 = \partial_1 d_2 \cdot d_1 = 0. \tag{1.7}$$

Mechanically, (1.7) can be interpreted as a no-bending condition within the plane of the strip. The cases $\delta_h \ll h^2$ and $\delta_h \sim h^2$ are still open even tough a candidate Γ-limit is known for $\delta_h \ll h^2$, as we see later.

The supercritical regime is analogous to the Von Kármán's energy scaling $\alpha > 2$ for plates. Three different models emerge, depending on the asymptotics behaviour of ε_h/δ_h^2. Roughly speaking, they correspond to the constrained, the standard, and the linearized Von Kármán's theory for plates. As it happens for plates, in the supercritical regime the quantity of interest is the displacement, since deformations approach the identity, up to a rotation and to a translation. Up to rescaling, the components $u_{h,1}$, $u_{h,2}$, and $u_{h,3}$ of the displacement are proved to converge to u_1, u_2, and u_3, having a precise structure:

(a) if $\varepsilon_h \ll \delta_h^2$ or $\varepsilon_h \sim \delta_h^2$, then u is a Bernoulli–Navier displacement, that is, u_2 and u_3 are independent of x_2 and x_3, while u_1 is affine in both x_2 and x_3, with coefficients given by $-\partial_1 u_2$ and $-\partial_1 u_3$, respectively,

(b) if $\varepsilon_h \gg \delta_h^2$, then $u_2 = 0$ and u_1, u_3 depend only on x_1 and satisfy the constraint

$$2\partial_1 u_1 + (\partial_1 u_3)^2 = 0. \tag{1.8}$$

Condition (1.8) is a one-dimensional version of (1.5). In the subcase $\varepsilon_h \gg \delta_h^2$, difficulties similar to the ones of the critical case are encountered, and the Γ-limit is computed only under the additional condition $h^2\varepsilon_h \ll \delta_h^2$.

The work of Chapter 5 is motivated by better understanding the regimes where the identification of the Γ-limit is still lacking. Heuristically, the two conditions $h^2 \ll \delta_h$ in the critical regime and $h^2\varepsilon_h \ll \delta_h^2$ in the supercritical regime mean that the thickness δ_h is not so small with respect to the width h, so that the ribbon behaves not so differently from a rod. Instead, when the thickness is much smaller that the width (e.g., when $\delta_h \ll h^2$ in the critical scaling), we expect the ribbon to behave more similarly to a plate. As a first approximation, one may start from a two-dimensional model on a thin strip $\Omega_h := (0, L) \times (-h/2, h/2)$, where the thickness is completely neglected. Its Γ-limit as $h \to 0$ should possibly provide insights on the missing cases in [FMP12; FMP13].

In [Fre+15], the Kirchhoff–Love model for plates is considered as starting point. The Γ-limit is a corrected Sadowsky's model, as it coincides with the relaxation of the model proposed by Sadowsky in [Sad30] (see also [HF16] for an English translation), to describe the optimal shape of a Möbius band at rest (see also [Fre+22; Fre+16]). This functional is expected to coincide with the Γ-limit in the critical regime when $\delta_h \ll h^2$.

In [Fre+17], starting from two-dimensional Von Kármán's models on the strip Ω_h, a full characterization of their Γ-limits as $h \to 0$ is shown. For the Von Kármán's model and the linearized model, the Γ-limit coincides with the one computed in [FMP13] in the supercritical regime when $\varepsilon_h \sim \delta_h^2$ and $\varepsilon_h \ll \delta_h^2$, respectively. For the constrained Von Kármán's model instead, a different Γ-limit of Sadowsky's type is found, that should correspond to the missing subcase $\delta_h^2 \ll h^2\varepsilon_h$.

In this thesis we study an intermediate two-dimensional energy, with the hope of shedding light on the missing Γ-limits in the three-dimension to one-dimension convergence. For a thin strip $\Omega_h := (0, L) \times (-h/2, h/2)$ we introduce the two-dimensional energy

$$\int_{\Omega_h} |\nabla w^T \nabla w - \mathrm{Id}\,|^2\,dx + \delta_h^2 \int_{\Omega_h} |\nabla^2 w|^2\,dx,$$

that now depends on the thickness parameter δ_h. As it is customary, we consider the energy per unit volume, and we rescale the strip to $\Omega := \Omega_1$, writing the energy in terms of the rescaled deformation

$$E_h(y) := \int_\Omega |\nabla_h y^T \nabla_h y - \mathrm{Id}\,|^2\,dx + \delta_h^2 \int_\Omega |\nabla_h^2 y|^2\,dx. \tag{1.9}$$

The mechanical interpretation of this energy is clear: the first term penalizes stretching whereas the second penalizes bending.

Energy (1.9) is frequently used in the physical and engineering literature. It can be seen as an expansion of the three-dimensional energy with respect to the thickness parameter. Indeed, the results recalled in in the previous sections show that the stretching term is of order one in the thickness, whereas the bending term is of order δ_h^2.

In Chapter 5, we study the Γ-convergence of $\delta_h^{-\alpha} E_h$ for $\alpha \geq 2$. These scalings correspond exactly to the critical ($\alpha = 2$) and the supercritical ($\alpha > 2$) regimes considered in [FMP12; FMP13].

Our first result is that the Γ-limit of $\delta_h^{-2} E_h$ depends on the asymptotic behaviour of δ_h / h^2, as expected. When $\delta_h \gg h^2$ we obtain the same model of [FMP12] (see Theorem 5.2.2), whereas when $\delta_h \ll h^2$ we show the Γ-convergence to the corrected Sadowsky's model mentioned above (see Theorem 5.2.7).

Let us briefly clarify the role of the ratio δ_h / h^2. The quantity δ_h / h^2 is linked with the behaviour of $\det(\nabla_h^2 y)$, that roughly speaking represents the Gauss' curvature of the deformed strip $y(\Omega)$. Recall that for an isometric immersion, the Gauss' curvature is zero, and so is $\det(\nabla_h^2 y)$. It is clear by the form of the energy $\delta_h^{-2} E_h$ (see (1.9)) that the faster δ_h is going to zero, the closer y should be to an exact isometric immersion of the strip. This phenomenon can be quantified by means of the following estimate:

$$\| \det(\nabla_h^2 y_h) \|_{L^1} \leq C \frac{\delta_h}{h^2}, \tag{1.10}$$

for a sequence y_h such that $E_h(y_h) \leq C \delta_h^2$. In particular, when $\delta_h \ll h^2$, the quantity $\det(\nabla_h^2 y_h)$ converges to zero. Instead, when $\delta_h \gg h^2$, the energy fails to provide any bound on $\det(\nabla_h^2 y_h)$. In other words, when $\delta_h \ll h^2$, deformations are so close to isometric immersions that the Γ-limit coincides with the one computed from the Kirchhoff's functional. The relaxation of the constraint $\det(\nabla_h^2 y) = 0$ leads to the corrected Sadowsky's model.

In Section 5.3 we discuss the Γ-convergence of $\delta_h^{-\alpha} E_h$ for $\alpha > 2$. We identify three different regimes:

(i) the constrained Von Kármán's regime $2 < \alpha < 4$,

(ii) the Von Kármán's regime $\alpha = 4$,

(iii) the linearized regime $\alpha > 4$.

In the last two cases, we show that the Γ-limits coincide with the one obtained in [FMP13].

When $\alpha \in (2, 4)$, we split the analysis depending on the asymptotics of $\delta_h^{2-\gamma} / h^2$, where $\gamma := \alpha/2$, in agreement with the regimes identified in [FMP13]. As in the scaling $\alpha = 2$, the behaviour of $\delta_h^{2-\gamma} / h^2$ is linked with a suitable rescaling of $\det(\nabla_h^2 y)$. In both cases $\delta_h^{2-\gamma} \ll h^2$ and $\delta_h^{2-\gamma} \gg h^2$ we find that the Γ-limits coincide with the ones obtained in [Fre+17] and [FMP13], respectively.

Our results show that the intermediate energy E_h is a good candidate to understand the Γ-limit behaviour of I_h for different scalings. Unfortunately, at the present time we are not able to identify the Γ-limit when $\delta_h \sim h^2$ for the energy $\delta_h^{-2} E_h$ and when $\delta_h^{2-\gamma} \sim h^2$ for the energy $\delta_h^{-\alpha} E_h$. These regimes are completely open and no candidate Γ-limit is known. In these cases, estimates of the type (1.10) are still enough to guarantee convergence of the Gauss' curvature, however, to a limit possibly different from zero. Further comments on the specific difficulties we encounter are highlighted at the end of Section 5.2.

The content of Chapter 5 is part of an ongoing project [MT25] in collaboration with M. G. Mora, inspired by some preliminary computations contained in [Fre+25].

1.2 Introduction to part II

Under small tensile load most solid materials show elastic behaviour: once the load is removed, the body goes back to its original shape. However, when the strain reaches a critical threshold—the yielding point—the deformation becomes irreversible and plastic phenomena intervene. The mathematical modeling of plasticity is a subject of ongoing debate and research, where multiple models and theories have been proposed. Without claiming to be exhaustive, we refer to [Lub08; Hil98] for an extended treatment. For metals, it is widely accepted in the mechanical literature that plastic effects are the macroscopic result of both the emergence and the motion of dislocations—microscopic defects in the crystalline atomic structure. In this manuscript, we focus on this latter class of materials.

In a two-dimensional setting, assuming the crystal lattice of a metal to be perfectly square, we can envision the atoms arranged along parallel lines. An edge dislocation is the defect produced by the presence of an extra half-line of atoms. The microscopic presence of an edge dislocation is usually described by the so-called Burgers' vector, that describes the amount and the direction of the atomic slip. In this setting, the Burgers' vector is defined as the vector needed to close a loop around the defect. More precisely, imagine drawing a loop around the defect in the crystal lattice, and then to draw the same circuit in a perfect reference crystal. In the presence of a defect, the new loop is not closed, and the missing vector needed to close the loop is exactly the Burgers' vector (see Figure 1.2). Note that the Burgers' vector always lies in the lattice itself.

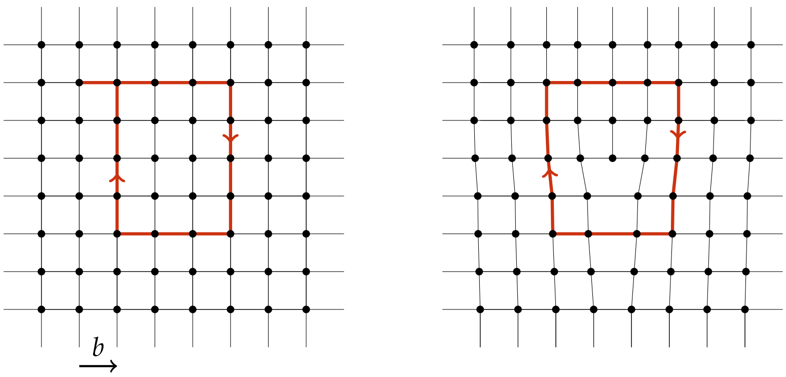

Figure 1.2: The Burger's vector b

The problems addressed in this thesis are framed in a semi-discrete setting, where the metal is treated as a continuum, averaging out its crystalline structure, whereas dislocations are still modelled as point defects.

In Chapter 7, we consider a d-dimensional extension of a two-dimensional non-local interaction problem for edge dislocations, and we provide a characterization of the ground states of the corresponding energy.

In Chapter 8, we change framework, and we focus on the emergence of the so-called grains. It is empirically observed that the crystal structure of a metal after a plastic deformation can be divided in regions, called grains, where the lattice has different orientations. The dislocations tend to accumulate on the boundary between these regions—the grain boundary. Starting from a model introduced in [LL16] we propose a grain boundary construction, and we show that its energy scaling agrees with the empirical one conjectured in [RS50].

Nonlocal interaction of dislocations and other particles

Suppose to have a two-dimensional isotropic crystal lattice and assume that all dislocations have the same Burger's vector, say e_1. Following computations from [Mor24, Section 1.1] and [HL82, Chapter 13–4], one can show that in a semidiscrete setting the force experienced by a dislocation located at a point $x \in \mathbb{R}^2$ due to the presence of another dislocation at the origin is of the form $F(x) = -(c\nabla\mathcal{W}(x), 0)$ for some material constant $c > 0$, where

$$\mathcal{W}(x) := -\log|x| + \frac{x_1^2}{|x|^2}.$$

In this context F is called the Peach–Köhler force, and $\mathcal{W}$ can be regarded as its potential energy. Assuming to have two dislocations at points $x, y \in \mathbb{R}^2$, their interaction energy is then

$$\mathcal{W}(x-y) = -\log|x-y| + \frac{(x_1-y_1)^2}{|x-y|^2}.$$

The first term of the energy is repulsive and would prefer the dislocations to be as far as possible, while the second term favours the vertical alignment of dislocations.

Considering n dislocations at the points $x_1, \dots, x_n$ and letting $n \to \infty$, it can be shown (see, for example, [Lan72, Point 2.3.12] or [BHS19, Section 4.2]) that the rescaled discrete energies

$$\frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathcal{W}(x_i - x_j)$$

Γ-converge in a suitable topology to the continuous energy

$$I(\mu) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{W}(x-y)\, d\mu(x)\, d\mu(y),$$

where μ is a probability measure.

A classical conjecture in plasticity is that at equilibrium dislocations having all e_1 as Burgers' vector should pile up vertically in a wall-like structure (see [DŠ04; LBN93; AHL16]). In our framework, this would amount to show that minimizers of I have a one-dimensional and vertical support. Note, however, that because of the repulsive nature of $\mathcal{W}$, existence of minimizers can be granted only if some confinement term is included in the energy.

A first result in this direction is obtained in [MRS18]. There, the dislocations are assumed to attract each other quadratically. The authors show that the minimizer of the interaction energy I augmented by a quadratic confinement term is the so-called semicircle law, a measure supported on a segment.

Since then, further generalizations of this result in various directions have been proved. Firstly, a tuning of the anisotropy term by a coefficient α has been added in [Car+19]. Then, the same problem has been considered in [MS21] where the quadratic confinement is replaced by the constraint that measures must have support inside a two-dimensional ellipse E.

Lastly, noting that the logarithmic kernel is the two-dimensional Coulomb's kernel, the problem has been considered in $\mathbb{R}^d$ with $\mathcal{W}$ replaced by an anisotropic Riesz' kernel of the form

$$\mathcal{W}_s(x) := \frac{1}{|x|^s}\Phi\Big(\frac{x}{|x|}\Big),$$

where $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ is the anisotropic profile and $s \in (0, d)$. Note that the Coulomb's kernel is retrieved when $s = d - 2$, while the logarithmic kernel by a suitable limit as $s \to 0$. This extension to higher dimensions has been treated in a series of works where the confinement is always assumed to be quadratic. Without claiming to be exhaustive, we recall [CS22; CS23a; CS23b; Mat+23a; Mat+23b; Fra+25].

In Chapter 7, we contribute to this line of research by characterizing the minimizer of

$$I_s(\mu) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathcal{W}_s(x-y)\, d\mu(x)d\mu(y), \qquad \mu \in \mathcal{P}(E)$$

over the set of probability measures μ with support contained in a prescribed d-dimensional ellipsoid E.

When Φ is constant, the interaction is isotropic and the study of minimizers of I_s is a classical problem in potential theory. If E is a compact set, one can show that the minimizer exists and is unique. The reciprocal of the minimal energy is the s-capacity of E (see [Lan72, Chapter II], where the s-capacity is called $(d-s)$-capacity). The explicit characterization of the minimizer $\mu_{\mathrm{iso},s}$ of I_s for Φ constant and E given by a ball dates back to the works of Szegő, Pólya, and Riesz [SP31; Rie88a; Rie88b]. A renovated interest brought new and different proofs, see [DKK16]. For E given by the ball B_1 centred at 0 with radius 1, the expression of the minimizer is

$$\mu_{\mathrm{iso},s} := \begin{cases} c_{s,d}(1-|x|^2)^{\frac{s-d}{2}}\mathcal{L}^d(x) \llcorner B_1 & \text{if } d-2 < s < d, \\ c_{s,d}\mathcal{H}^{d-1} \llcorner \partial B_1 & \text{if } 0 < s \le d-2, \end{cases}$$

where $c_{s,d}$ is a normalization constant (see Lemma 7.3.1 for its exact value). The value $s = d-2$, corresponding to the Coulomb's kernel, acts as a threshold: in the super-Coulombic regime the minimizer is absolutely continuous with respect to the Lebesgue measure and is supported on the whole ball, whereas in the sub-Coulombic regime it becomes singular, and its support reduces to the boundary.

In Chapter 7 we consider a rather general anisotropic profile Φ, and we show that the minimizer of I_s exists, and it is unique provided the Fourier transform $\widehat{\mathcal{W}_s}$

is nonnegative. Under the same assumption, we prove that the minimizer can be characterized by the following Euler–Lagrange equations (see Theorem 7.2.1):

$$\operatorname{supp}\mu \subset E, \tag{EL1}$$
$$(\mathcal{W}_s * \mu)(x) = C \quad \text{for } \mu\text{-a.e. } x \in \operatorname{supp}\mu, \tag{EL2}$$
$$(\mathcal{W}_s * \mu)(x) \geq C \quad \text{for every } x \in E \setminus N \text{ with } \operatorname{Cap}_s(N) = 0, \tag{EL3}$$

for some constant C, where Cap_s is the s-capacity. Note that these conditions are classical in potential theory, and are consistent with those of the isotropic case.

Under the condition $\widehat{\mathcal{W}_s} \geq 0$, we show that in the super-Coulombic regime, the minimizer of I_s is the push-forward of $\mu_{\mathrm{iso},s}$ onto E. In particular, the anisotropy plays no role in the optimal distribution, which is identified solely by the confinement term. This is a surprising result, that is however consistent with the one already proved in dimension two for the logarithmic kernel in [MS21].

The strategy of the proof is rather simple, and it amounts to show that the push-forward μ_s^E of $\mu_{\mathrm{iso},s}$ onto E solves the Euler–Lagrange equations. To prove (EL2), we derive a novel integral formula for the potential function $\mathcal{W}_s * \mu_s^E$ in terms of the Fourier's transform of $\mathcal{W}_s$ (see Theorem 7.3.6). This idea was already used in [Fra+25], where however the corresponding formula for the potential function was proved only for $s \in (0,5] \cap [d-3, d)$. Here, instead, using a different approximation technique, we manage to treat any Riesz exponent $s \in (0, d)$ in any dimension d. We remark that the formula we derive is valid only inside E. This is enough for our purposes, but it is not sufficient to extend to any dimension the results proved in the case of a quadratic confinement (see [Fra+25]).

Lastly, we discuss the sub-Coulombic regime $s < d-2$, and we show that in this case the anisotropy may change the optimal distribution. To do so, we restrict ourselves to the easier case $E = B_1$, and we provide an explicit example of anisotropy Φ for which $\mu_{\mathrm{iso},s}$ is not the minimizer (see Section 7.4).

The main results of Chapter 7 can also be found in [Mor+25].

Construction of grain boundaries

In the last chapter of this thesis we shift our attention to a semi-discrete model for grain boundaries proposed in [LL16], and we consider a two-dimensional rectangular section $\Omega \subset \mathbb{R}^2$ of a crystal. As dislocations are microscopic defects, we introduce a small parameter $\varepsilon > 0$, representing the small size of the lattice cell. We consider τ and λ, two positive parameters representing the rescaled Burgers' vector length and the rescaled size of the core region around a dislocation, respectively. Lastly, we introduce a Bravais' lattice $\mathcal{B}$, corresponding to the crystalline structure of the material. We remark that, in semi-discrete models, there is coexistence of both macroscopic and microscopic scales. Thus, even if the material is represented as a continuum, the fact that it has a crystalline structure is somehow taken into account by the quantities ε, λ, and τ and the Bravais' lattice $\mathcal{B}$.

Heuristically, at a distance from the defects comparable to the lattice spacing, the material's crystalline structure should locally resemble the reference lattice after a distortion. Conversely, the continuum approximation should break down close to the defects. The model proposed in [LL16] describes exactly this situation.

We may divide Ω in two regions: $B_{\lambda\varepsilon}(S)$, where the dislocations concentrate and $\Omega \setminus B_{\varepsilon\lambda}(S)$, where instead the material is a local distortion of a perfect lattice. Here, $S \subset \Omega$ is a closed set and $B_{\varepsilon\lambda}$ is a tubular neighbourhood of S of radius $\lambda\varepsilon$.

The state space is given by pairs (β, S), with $\beta : \Omega \to \mathbb{R}^{2\times 2}$ representing the strain field. In order for the pair (β, S) to be admissible, some conditions are required. Firstly, the support of $\operatorname{curl}\beta$ should be contained in S. Since $\operatorname{curl}\beta$ represents the macroscopic Burgers' vector, this condition grants that the defects are contained in S. The second condition is called quantization of the averaged Burger's vector. If β is sufficiently regular, it can be written as

$$\int_{\partial\Gamma} \beta t \, dx = \int_{\Gamma} \operatorname{curl}\beta \, dx \in \tau\varepsilon\mathcal{B},$$

where Γ is any sufficiently regular set whose boundary does not cross $B_{\lambda\varepsilon}(S)$ and t is the tangent vector to $\partial\Gamma$, suitably oriented. Taking $\partial\Gamma$ to be a loop around a defect, this condition implies that the Burgers' vector associated with the defect is a vector of the scaled lattice $\varepsilon\tau\mathcal{B}$. In particular, the microscopical slip that it represents must be in the admissible directions of the lattice and of a size that is a multiple of the rescaled lattice spacing $\tau\varepsilon$.

The energy of an admissible pair (β, S) has the form

$$E_\varepsilon(\beta, S) := \frac{1}{\varepsilon}\Big(\int_{\Omega\setminus B_{\lambda\varepsilon}(S)} \mathcal{W}(\beta) \, dx + \mathcal{L}^2(B_{\lambda\varepsilon}(S)) \Big),$$

where $\mathcal{L}^2$ is the two-dimensional Lebesgue measure. The first term of the energy is the elastic contribution, whereas the second, also called core energy, is related to presence of the defects.

Some words should be spent on the scaling coefficient $1/\varepsilon$. When applying incompatible boundary conditions, e.g., opposite rotations on the lateral sides of the rectangle Ω, we expect the emergence of grains, that is, regions where the lattice has different orientations. The energy needed for the formation of the grains should be concentrated on the common interface—the grain boundary. When $\varepsilon \to 0$, the first term of the energy forces β to be close to the set of zeros of $\mathcal{W}$, whereas the second term should force the set S to approximate a one-dimensional interface, so that $\mathcal{L}^2(B_{\lambda\varepsilon}(S)) \sim \varepsilon$. This heuristic argument has been recently made rigorous in [FGS25], where the authors show the Γ-convergence of E_ε to an interfacial energy, as $\varepsilon \to 0$.

The simplest setting in which the emergence of a grain boundary is expected is when the vertical sides of the rectangle Ω are rotated of opposite and small angles $\pm\alpha$ (see Figure 1.3). When the lattice is the standard square lattice, a vertical grain boundary should appear separating two regions where the lattice is almost perfectly rotated. In the case of two non-symmetric rotations at the boundary, one can simply rotate the body so that the rotations become symmetric, at the cost of accounting for a rotated Bravais' lattice. This is exactly the setting of [RS50], where the authors formally derive a model for the creation of such interfaces, and show the agreement of their prediction with experimental data. In particular, they estimate an energy of order $\alpha|\log\alpha|$ for the emergence of a grain-boundary.
In [FGS25], the authors show the interfacial energy arising as the Γ-limit of E_ε satisfies the latter scaling in α.

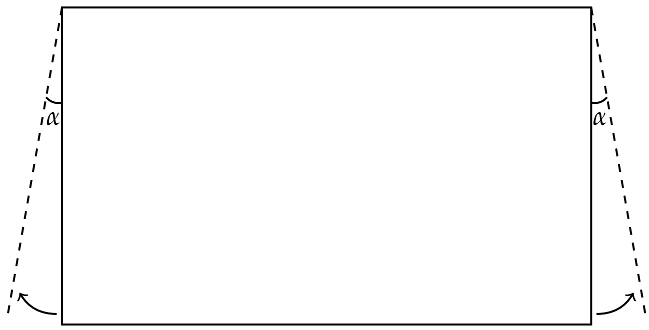

Figure 1.3: The rectangle Ω with the imposed boundary condition.

In Chapter 8, we give a simpler and more physical proof of the scaling predicted by [RS50]. Precisely, we show that the minimum of the energy E_ε under opposite and symmetric rotations of the vertical boundaries is bounded from above by a term of order $\alpha|\log\alpha|$. The result holds for a general lattice and with any orientation, covering also the case of non-symmetric boundary conditions. To obtain this upper bound, we explicitly construct a field β simulating the grain boundary, and we show that its energy is of order $\alpha|\log\alpha|$.

As this result was already obtained in [FGS25, Section 5.2], we should remark the main differences of our approaches. The strain field β we construct is piecewise constant on a finite Cacciopoli's partition of Ω, and thus is associated with a piecewise affine deformation at ε distance from the defects. Instead, the deformation constructed in [FGS25] contains nonlinearities of the form x_2/x_1 and x_1/x_2, whose physical meaning is unclear. In the region approximating the grain boundary, our construction alternates in a periodic way dislocations simulating slips in the two admissible directions of the lattice. The same strategy is used in [FGS25], albeit with twice the number of dislocations associated with one of the two Burgers' vector direction.

We believe that our approach is simpler, both in the construction and in the resulting field, leading to greater possibilities of generalization.

The content of Chapter 8 is part of an ongoing project [ST25] in collaboration with L. Scardia.

I

Dimension reduction problems

2

Notation and mathematical preliminaries

2.1 Notation and general assumptions

Low dimensional bodies. Throughout the first part of the thesis, $h > 0$ denotes the small dimension of a physical object. In the following, we treat two types of low dimensional bodies: plates—a three-dimensional body with small thickness—and strips, whose profile is fundamentally one-dimensional and that we represent as two-dimensional rectangles with small width. In both cases, the geometry of the reference configuration is flat, with no prior curvature imposed to the system. In other words, there is no prestrain. Thus, when we consider plates, h represents their thickness, whereas when we consider strips, h represents their width.

The letter S is used to denote the lower dimensional object that we use to describe the limiting behaviour: the mid-plane for plates or the mid-line for strips. We use Ω_h to denote the reference configuration of the full dimensional object, namely a cylinder with height h and base S. The set Ω_h is always defined so that it is symmetric with respect to the last variable: the small dimension. For example, a plate with thickness h is $\Omega_h = S \times (-h/2, h/2)$, with S a suitable subset of $\mathbb{R}^2$.

Elastic energy. A deformation of Ω_h is described by a map $w : \Omega_h \to \mathbb{R}^3$ sending each point $x \in \Omega_h$ to its new position $w(x)$. Its elastic energy is induced by a density $\mathcal{W}$ and is given by

$$\int_{\Omega_h} \mathcal{W}(\nabla w)dx.$$

The elastic energy density is defined either on $\mathbb{R}^{3\times 2}$ or $\mathbb{R}^{3\times 3}$, depending on whether we are working with strips or plates, and takes values in $[0, +\infty]$. We assume that $\mathcal{W}$ is a Borel measurable function such that

(RG) $\mathcal{W}$ is C^2 in a neighbourhood of the set $K_{\mathcal{W}} := \{\mathcal{W} = 0\}$, (FI) $\mathcal{W}$ is frame indifferent, i.e., $\mathcal{W}(RM) = \mathcal{W}(M)$ for every $R \in \mathrm{SO}(3)$ and for every $M \in \mathbb{R}^{3\times 3}$ or $M \in \mathbb{R}^{3\times 2}$, according to the context.

While the first condition is needed from a mathematical standpoint, the second one, usually known as frame indifference, is physically motivated and describes the invariance of the energy with respect to rigid changes of the reference frame.

In Chapter 5, where we deal with strips, the energy $\mathcal{W}$ has an explicit expression, satisfying both (RG)–(FI).

Dealing with plates, in Chapters 3 and 4, we allow for more general densities $\mathcal{W}$, defined on $\mathbb{R}^{3\times 3}$. In the latter case, the set of minimizers of the energy density $K_{\mathcal{W}}$ is made of a finite number of so-called wells, centred at $U_1, \dots, U_l \in R^{3\times 3}$. Precisely,

$$K_{\mathcal{W}} = \bigcup_{i=1}^{l} \mathrm{SO}(3) U_i, \tag{2.1}$$

where each set $K^i_{\mathcal{W}} := \mathrm{SO}(3)U_i$ is a well. Note that, by (FI), if $U_i \in K_{\mathcal{W}}$ then $\mathrm{SO}(3)U_i \subset K_{\mathcal{W}}$. We assume the matrices $U_1, \dots, U_l$ to be invertible, with positive determinant, and such that $U_i^{-1}U_j \notin \mathrm{SO}(3)$ for every $i \neq j$. The last condition grants that the wells $K^i_{\mathcal{W}}$ are pairwise disjoint. By polar decomposition and (FI) we can further assume without loss of generality $U_1, \dots, U_l$ to be symmetric and positive definite. Whenever there is no ambiguity between different elastic energy densities, we write K in place of $K_{\mathcal{W}}$. We say that $\mathcal{W}$ has a single-well structure if $l = 1$. In that case, we assume that $U_1 := \mathrm{Id}$. If $l > 1$ we say that $\mathcal{W}$ has a multi-well structure.

Depending on the mathematical application we pursue, we assume different growth behaviour of $\mathcal{W}$ outside $K_{\mathcal{W}}$. The precise hypotheses we need are postponed to the corresponding chapters.

Rescaled variables and energy per unit volume. Since the reference configuration Ω_h depends on h, it is usually preferred to rewrite the elastic energy in terms of the rescaled reference configuration $\Omega := \Omega_1$. To do so, we introduce the rescaled deformation, that we denote by y and its rescaled gradient, that we write as $\nabla_h y$. Precisely, y is given by the relation

$$y(x,z) := w(x,hz), \qquad (x,z) \in S \times (-1/2, 1/2) =: S \times I,$$

while the rescaled gradient is defined as

$$\nabla_h y := \begin{pmatrix} \nabla_x y & \frac{1}{h}\partial_z y \end{pmatrix}.$$

By a simple change of variable, the energy takes the form

$$h \int_\Omega \mathcal{W}(\nabla_h y)\, dx.$$

We denote by I_h the energy per unit volume, that is the elastic energy divided by h

$$I_h(y) := \int_\Omega \mathcal{W}(\nabla_h y)\, dx.$$

With some abuse of language, we say that I_h is the elastic energy.

Matrices and vectors. Throughout the first part of the thesis, we use some matrix notation that we introduce now. The vectors $\{e_j\}_{j=1,\dots,n}$, represent the standard basis of $\mathbb{R}^n$. We denote by $\mathbb{R}^{n\times n}_{\text{sym}}$ and $R^{n\times n}_{\text{skew}}$ the spaces of symmetric and skew-symmetric $n \times n$ matrices, respectively. We use $\mathrm{Id}_{3\times 2}$ to denote

$$\mathrm{Id}_{3\times 2} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As it is customary, we define

$$\begin{aligned} \mathrm{SO}(n) &:= \{M \in \mathbb{R}^n : M^T M = \mathrm{Id} \text{ and } \det(M) = 1\}, \\ \mathrm{O}(n) &:= \{M \in \mathbb{R}^n : M^T M = \mathrm{Id}\}. \end{aligned}$$

For $n, k \in \mathbb{N}$ with $n > k$, we write $\mathrm{O}(n,k)$ to denote

$$\{M \in \mathbb{R}^{n\times k} : M^T M = \mathrm{Id}_{k\times k}\}.$$

Given a matrix $M \in \mathbb{R}^{3\times 3}$ or $M \in \mathbb{R}^{3\times 2}$, we write M' to denote the top-left 2×2 submatrix. For a vector $v \in \mathbb{R}^3$ we write v' in place of (v_1, v_2). Similarly, we write ∇' to denote $\begin{pmatrix}\partial_1 & \partial_2\end{pmatrix}$. When we are working with plates, i.e., S is a subset of $\mathbb{R}^2$, we write ∇' in place of ∇ even for functions defined on the mid-plane S, for which $\nabla = \nabla'$, in order to stress the dimensionality of the gradient.

We use the super(sub)scripts to denote submatrices of $M \in \mathbb{R}^{3\times 3}$ in the following way: every missing subscript index is a removed row while every missing superscript index is a removed column. For example, $M^{1,2}$ is the 3×2 submatrix given by the first two columns of M while $M_{1,2}$ is the 2×3 submatrix given by the first two rows of M.

Whenever we sum or multiply matrices and vectors with different dimension we imply that the smaller one is naturally embedded in the bigger space by adding zeros in the missing entries. For example, if $M \in \mathbb{R}^{2\times 2}$ and $A \in \mathbb{R}^{3\times 3}$ the expression $M + A$ means $\iota(M) + A$ where

$$\iota : \mathbb{R}^{2\times 2} \hookrightarrow \mathbb{R}^{3\times 3}, \quad M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}.$$

Given two vectors $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^n$ we denote by $u \otimes v \in \mathbb{R}^{k\times n}$ the matrix

$$(u \otimes v)_{ij} = u_i v_j.$$

For a tensor $\mathbb{B} \in \mathbb{R}^{3\times 2\times 2}$, we define

$$\det(\mathbb{B}) := \sum_{j=1}^{3} (\mathbb{B}_{j11}\mathbb{B}_{j22} - \mathbb{B}^2_{j12}).$$

Big O and small o notation. We often employ the big-O and small-o notation. Given two sequences (a_h), $(b_h) \subset \mathbb{R}$, recall that we write $b_h = O(a_h)$ whenever, at least for $h \ll 1$, $|b_h| \leq C|a_h|$ for some constant $C > 0$ independent of h, whereas we write $b_h = o(a_h)$ whenever $b_h/a_h \to 0$ as $h \to 0$.

Given three sequences (a_h), (b_h), $(c_h) \subset \mathbb{R}$, we write $c_h = O(a_h, b_h)$ meaning that there exists a positive constant $C > 0$ independent of h such that $|c_h| \leq C(|a_h| + |b_h|)$. In particular, if $|a_h| \ll |b_h|$ for $h \to 0$, we have $O(a_h, b_h) = O(b_h)$.

Unless otherwise stated, when we consider a sequence of functions f_h and we write $f_h = O(a_h)$ we are tacitly assuming that the constant C can be chosen uniformly in x. Similarly, the convergence $f_h/a_h \to 0$ is implicitly assumed to be uniform. If the functions f_h are smooth, the constant C (and the convergence for the small o notation) is assumed to be uniform in x and in its derivatives.

Isometric immersions. Suppose that $S \subset \mathbb{R}^2$ is an open set. We say that a deformation $w : S \to \mathbb{R}^3$ is an isometric immersion—with respect to the Euclidean metric—if

$$\nabla' w^T \nabla' w = \mathrm{Id}\,.$$

We denote by $W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$ the space of Sobolev isometric immersions, namely

$$W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3) := \{y \in W^{2,2}(S;\mathbb{R}^3) : \nabla' y^T \nabla' y = \mathrm{Id} \text{ a.e. in } S\}.$$

More generally, given a flat metric represented by a 2×2 positive-definite constant matrix $\mathcal{G}$, we say that $w : S \to \mathbb{R}^3$ is an isometric immersion with respect to $\mathcal{G}$ if

$$\nabla' w^T \nabla' w = \mathcal{G}.$$

Analogously to the Euclidean case, we define

$$W^{2,2}_{\mathrm{iso},\mathcal{G}}(S;\mathbb{R}^3) := \{y \in W^{2,2}(S;\mathbb{R}^3) : \nabla' y^T \nabla' y = \mathcal{G} \text{ a.e. in } S\}.$$

Dead loads and optimal rotations. In some of our results, we account for the presence of some dead loads acting on the body. We use f_h to denote a sequence of loads acting on S, that is, a sequence of maps $f_h : S \to \mathbb{R}^3$ representing the density of some forces. The sequence (f_h) converges in some suitable sense—that we specify later depending on the application we have in mind—to some limit load f.

In various results of Chapters 3 and 4, we observe that the forces acting on S select some preferred minimizers of $\mathcal{W}$. These special minimizers are the generalization to our framework of the optimal rotations introduced in [MM21]. We denote by $\mathcal{M}_h \subset K_{\mathcal{W}}$ the set of maximizers of the functional

$$F_h \colon K_{\mathcal{W}} \to \mathbb{R}, \quad RU_j \mapsto \int_S f_h \cdot RU_j \begin{pmatrix} x' \\ 0 \end{pmatrix} dx.$$

A maximizer of F_h represents a trivial deformation with zero elastic energy that minimizes the work done by the force f_h. Note that $\mathcal{M}_h$ is not empty by compactness of $K_{\mathcal{W}}$. Similarly, we define

$$\mathcal{M} := \operatorname*{argmax}_{K_{\mathcal{W}}} F,$$

where

$$F(A) := \int_S f \cdot A \begin{pmatrix} x' \\ 0 \end{pmatrix} dx.$$

For $j = 1, \dots, l$, we define the sets $\mathcal{R}^j$ and $\mathcal{R}^j_h$ as follows

$$\mathcal{R}^j := \operatorname*{argmax}_{R \in \mathrm{SO}(3)} F(RU_j),$$
$$\mathcal{R}^j_h := \operatorname*{argmax}_{R \in \mathrm{SO}(3)} F_h(RU_j).$$

The elements of the sets $\mathcal{R}^j$ and $\mathcal{R}^j_h$ are called optimal rotations with respect to the well centred at U_j. By (2.1), there are subsets of indices $\Lambda_h, \Lambda \subset \{1, \dots, l\}$ such that

$$\mathcal{M}_h = \bigcup_{j \in \Lambda_h} \mathcal{R}^j_h U_j,$$
$$\mathcal{M} = \bigcup_{j \in \Lambda} \mathcal{R}^j U_j.$$

One can prove that $\mathcal{R}^j_h$ and $\mathcal{R}^j$ are closed, connected, boundaryless, and totally geodesic submanifolds of $\mathrm{SO}(3)$. Indeed, the proof of [MM21, Proposition 4.1] does not rely on the specific structure of F or F_h, but only on their linearity. We denote by $T\mathcal{R}^j_R$ and $N\mathcal{R}^j_R$ the tangent space and the normal space to $\mathcal{R}^j$ at the point R, respectively. By [MM21, Proposition 4.1], we have

$$T\mathcal{R}^j_R = \left\{ RW \in \mathbb{R}^{3\times 3} \colon W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}},\ F(RW^2U_j) = 0 \right\}, \tag{2.2}$$
$$N\mathcal{R}^j_R = \left\{ RW \in \mathbb{R}^{3\times 3} \colon W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}},\ RW \perp T\mathcal{R}^j_R \right\}. \tag{2.3}$$

Similarly, we define $T\mathcal{R}^j_{hR}$ and $N\mathcal{R}^j_{hR}$.

Recall that a geodesic in $\mathrm{SO}(3)$ is a curve of the form $t \mapsto Re^{tW}$, where $R \in \mathrm{SO}(3)$ and $W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$. By [MM21, Lemma 4.4], every geodesic generating from a point $R \in \mathcal{R}^j$ in tangential direction, is contained in $\mathcal{R}^j$. Precisely,

$$R \in \mathcal{R}^j \implies Re^{tW} \in \mathcal{R}^j \qquad \forall\, W \in \mathrm{T}\mathcal{R}_R,\ \forall\, t \in \mathbb{R}. \tag{2.4}$$

The same result holds for $\mathcal{R}^j_h$.

We can define the projection operators P^j and P^j_h of $\mathrm{SO}(3)$ onto $\mathcal{R}^j$ and $\mathcal{R}^j_h$, respectively. These projections have to be understood with respect to the intrinsic distance of $\mathrm{SO}(3)$, i.e.,

$$\mathrm{dist}_{\mathrm{SO}(3)}(R, Q) = \min \left\{ |W| \colon W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}},\ Q = Re^W \right\}. \tag{2.5}$$

They are well-defined at least in a neighbourhood of $\mathcal{R}^j$ and $\mathcal{R}^j_h$, respectively.

If $\mathcal{W}$ has a single-well structure, then the latter notation can be greatly simplified, as we have $\mathcal{M} = \mathcal{R}^1$ and $\mathcal{M}_h = \mathcal{R}^1_h$. In the latter case we use the letter $\mathcal{R}$ to denote both $\mathcal{R}^1$ and $\mathcal{M}$.

2.2 Mathematical preliminaries

For the sake of streamlining the exposition of the next chapters and give more emphasis to the main arguments used therein, in this section we present some accessory results. Some of these are variants of well-known facts, others are technical statements, or slight generalization of them. The section is further divided in three parts: the first is concerned with some simple results about the elastic energy density $\mathcal{W}$, the second contains some technical results on isometric immersions, and the last regards optimal rotations.

We start by stating a non-standard version of the Poincaré's inequality, the Generalized Dominated Convergence Theorem and the Gronwall's Lemma.

Theorem 2.2.1. *Let $p \geq 1$ and let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and connected set with Lipschitz boundary. Let $E \subset \Omega$ be a measurable set of positive Lebesgue measure. Then, there is a constant C_P depending on $|E|$ such that*

$$\|u\|_{L^p} \leq C_P \|\nabla u\|_{L^p}, \qquad \text{for every } u \in W^{1,p}(\Omega) \text{ s.t. } u \equiv 0 \text{ on } E.$$

A slightly more general result, from which Theorem 2.2.1 follows, is proved in [Zie89, Theorem 4.4.2]. Note that the constant C_P may blow up as $|E| \to 0$.

Theorem 2.2.2 (Generalized Dominated Convergence Theorem)**.** *Let (f_k), $(g_k) \subset L^1(\Omega)$ be two sequences such that*

(i) *$f_k \to f$ almost everywhere,*

(ii) *$g_k \to g$ in $L^1(\Omega)$,*

(iii) *$|f_k| \leq g_k$ for every $k \in \mathbb{N}$.*

Then

$$\lim_{k\to\infty} \int_\Omega f_k \, dx = \int_\Omega f \, dx.$$

A proof can be found, for example, in [EG15, Theorem 4, p.21].

Lemma 2.2.3 (Gronwall's Lemma)**.** *Let η be a non-negative, absolutely continuous function on $[0, T]$, such that for almost every $t \in [0, T]$*

$$\partial_t \eta(t) \leq \phi(t)\eta(t) + \psi(t),$$

where ϕ, $\psi \in L^1(0, T)$ are non-negative. Then

$$\eta(t) \leq e^{C(t)} \Big(\eta(0) + \int_0^t \psi(s) \, ds \Big) \qquad \forall\, t \in [0, T],$$

where

$$C(t) := \int_0^t \phi(s) \, ds.$$

We refer to [Eva10, Appendix B.2] for a proof.

2.2.1 Results concerning the elastic energy density

In this section, $\mathcal{W}$ is defined on $\mathbb{R}^{3\times3}$ and satisfies the assumptions given in Section 2.1. We prove some simple results concerning symmetry properties of its

Hessian at minimizers. Moreover, we prove—under suitable growth conditions—that the following function is well defined:

$$\bar{Q}_j\colon \mathbb{R}^{2\times 2} \to \mathbb{R} \quad M \mapsto \min_{a\in\mathbb{R}^3} Q_j(U_j^{-1}(\mathrm{sym}(M) + a\otimes e_3 + e_3\otimes a))$$

where

$$Q_j(M) := \nabla^2\mathcal{W}(U_j)M\colon M, \qquad j = 1,\dots,l,$$

and $U_j \in K_{\mathcal{W}}$ are the centres of the wells. Indeed, to rigorously derive plate models, a linearization around a minimizer of the energy density is needed, and the quadratic forms $\bar{Q}_j$ play a crucial role.

Lemma 2.2.4. *Suppose that* $\mathrm{Id} \in K_{\mathcal{W}}$. *Then* $\nabla^2\mathcal{W}(\mathrm{Id})$ *is a fourth-order symmetric tensor, namely*

$$\nabla^2\mathcal{W}(\mathrm{Id})M\colon M = \nabla^2\mathcal{W}(\mathrm{Id})\,\mathrm{sym}(M)\colon\,\mathrm{sym}(M) \qquad \forall\, M \in \mathbb{R}^{3\times 3}.$$

Proof. Let $A \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ and define $\phi(t) := e^{tA}$ for $t \in [-1,1]$. Note that $\phi(t) \in \mathrm{SO}(3)$ for every $t \in [0,1]$ so that in particular

$$\mathcal{W}(\phi(t)) = 0 \qquad \forall\, t \in [-1,1]. \tag{2.6}$$

Since $\phi'(t) = Ae^{tA}$ and $\phi''(t) = A^2e^{tA}$, differentiating (2.6) we get

$$0 = \nabla^2\mathcal{W}(\phi(t))\nabla\phi(t)\colon \nabla\phi(t) + \nabla\mathcal{W}(\phi(t))\colon \nabla^2\phi(t) \qquad \forall\, t \in (-1,1). \tag{2.7}$$

Since $\phi(0) = \mathrm{Id}$ and $\nabla\mathcal{W}(\mathrm{Id}) = 0$, from (2.7) we deduce that

$$0 = \nabla^2\mathcal{W}(\mathrm{Id})A\colon A \qquad \forall\, A \in \mathbb{R}^{3\times 3}_{\mathrm{skew}},$$

concluding the proof. □

As a simple consequence, we deduce some symmetry properties of $\nabla^2\mathcal{W}$ at other minimizers.

Lemma 2.2.5. *Suppose that* $U \in K_{\mathcal{W}}$. *Let* $Q(M) := \nabla^2\mathcal{W}(U)M\colon M$. *Then*

$$Q(M) = Q(\mathrm{sym}(MU^{-1})U) = Q(U^{-1}\,\mathrm{sym}(UM)) \qquad \forall\, M \in \mathbb{R}^{3\times 3}.$$

Proof. Define $\widetilde{\mathcal{W}}(F) := \mathcal{W}(FU)$. Clearly, $\widetilde{\mathcal{W}}$ satisfies (RG) and (FI). Moreover, $\mathrm{Id} \in K_{\widetilde{\mathcal{W}}}$. Thus, by Lemma 2.2.4 we have

$$\nabla^2\widetilde{\mathcal{W}}(\mathrm{Id})M\colon M = \nabla^2\widetilde{\mathcal{W}}(\mathrm{Id})\,\mathrm{sym}(M)\colon\,\mathrm{sym}(M).$$

By some simple computation we get $\nabla^2\widetilde{\mathcal{W}}(\mathrm{Id})M\colon M = Q(MU)$. Hence,

$$Q(M) = Q(MU^{-1}U) = Q(\mathrm{sym}(MU^{-1})U).$$

To conclude, observe that by definition of symmetric part and the symmetry of U we have

$$Q(\mathrm{sym}(MU^{-1})U) = Q(\mathrm{sym}(U^{-1}UMU^{-1})U) = Q(U^{-1}\,\mathrm{sym}(UM)).$$

□

Lastly, we show some coercivity of $\nabla^2\mathcal{W}$ at minimizers, provided $\mathcal{W}$ satisfies suitable growth conditions.

Lemma 2.2.6. *Suppose that $U \in K_{\mathcal{W}}$. Assume that $\mathcal{W}$ satisfies the following growth condition:*

$$\mathcal{W}(M) \geq C\,\mathrm{dist}^2(M, K_{\mathcal{W}}), \qquad \forall\, M \text{ in a neighbourhood of } K_{\mathcal{W}}. \tag{2.8}$$

Then, there exists $\lambda > 0$ such that for every $M \in \mathbb{R}^{3\times 3}$

$$\nabla^2\mathcal{W}(U)U^{-1}\,\mathrm{sym}(M) : U^{-1}\,\mathrm{sym}(M) \geq \lambda|\,\mathrm{sym}(M)|^2.$$

Proof. Without loss of generality, by (2.1) we can assume that $U = U_j$ for some $j = 1, \dots, l$. Hence, by (2.8) we have, for $\varepsilon \ll 1$

$$\mathcal{W}(U_j + \varepsilon U_j^{-1}M) \geq C\,\mathrm{dist}^2(U_j + \varepsilon U_j^{-1}M, K_{\mathcal{W}}) = C\,\mathrm{dist}^2(U_j + \varepsilon U_j^{-1}M, K^j_{\mathcal{W}}).$$

By the Taylor expansion of $\mathcal{W}$ at U_j (recall that the energy density $\mathcal{W}$ satisfies (RG)), and by Lemma 2.2.5 we get

$$\begin{aligned}
&\varepsilon^2\nabla^2\mathcal{W}(U_j)U_j^{-1}\,\mathrm{sym}(M) : U_j^{-1}\,\mathrm{sym}(M) + o(\varepsilon^2) \geq C\,\mathrm{dist}^2(U_j + \varepsilon U_j^{-1}M, K^j_{\mathcal{W}})\\
&\quad \geq C\,\mathrm{dist}^2(\mathrm{Id} + \varepsilon U_j^{-1}MU_j^{-1}, \mathrm{SO}(3)) = C\varepsilon^2|\,\mathrm{sym}(U_j^{-1}MU_j^{-1})|^2 + o(\varepsilon^2)\\
&\quad = C\varepsilon^2|U_j^{-1}\,\mathrm{sym}(M)U_j^{-1}|^2 + o(\varepsilon^2) \geq C\varepsilon^2|\,\mathrm{sym}(M)|^2 + o(\varepsilon^2).
\end{aligned}$$

Dividing by ε^2 and passing to the limit as $\varepsilon \to 0$ we conclude. ☐

As a consequence of these properties, we can show that $\bar{Q}_j$ is well defined under the quadratic growth assumption (2.8).

Proposition 2.2.7. *Assume that $\mathcal{W}$ satisfies the growth condition (2.8). Then $\bar{Q}_j$ is well-defined. Moreover, the function*

$$L_j : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^3, \qquad M \mapsto \underset{v\in\mathbb{R}^3}{\mathrm{argmin}}\{Q_j(U_j^{-1}(M + v\otimes e_3 + e_3\otimes v))\} \tag{2.9}$$

is well-defined and linear.

Proof. The function is $\bar{Q}_j$ is well-defined by the coercivity property proven in Lemma 2.2.6. To show that L_j is well defined and linear, note that $L_j(M)$ is the unique vector $x \in \mathbb{R}^3$ that solves the linear optimality conditions

$$\mathbb{C}(M + x\otimes e_3 + e_3\otimes x) : (e_i\otimes e_3 + e_3\otimes e_i) = 0, \qquad i = 1, 2, 3,$$

where $\mathbb{C}$ is the fourth-order symmetric tensor representing the quadratic form $M \mapsto Q(U_j^{-1}M)$. ☐

We conclude this section by stating a useful property of frame indifferent energy densities.

Lemma 2.2.8. *It holds that*

$$\mathcal{W}(M) = \mathcal{W}\left(\sqrt{M^TM}\right) \qquad \forall\, M \in \mathbb{R}^{3\times 3} \text{ with } \det(M) > 0.$$

Proof. By polar decomposition, every matrix $M \in \mathbb{R}^{3\times 3}$ with $\det(M) > 0$ can be written as $M = R\sqrt{M^TM}$, for some $R \in \mathrm{SO}(3)$. Then, the result follows by (FI). ☐

2.2.2 Results concerning isometries

In this section, we collect some statements regarding isometries. We recall and refine some results contained in [FJM06, Section 8], allowing us to construct isometric immersions with some prescribed data. Then, we report a density result for smooth isometric immersions proved in [Hor11], and we show a new density statement for solutions of the Monge–Ampère equation that is of interest on its own. In the last part, we state a result from [Fre+16] concerning the costruction of isometric immersions of thin strips, whose restriction to the mid-line is known.

For the first part of the section, the set S represents the mid-plane of a plate, thus we assume $S \subset \mathbb{R}^2$ to be an open, bounded, and connected set with Lipschitz boundary.

Consider a function $v \in W^{2,2}(S)$. We would like to construct an isometric immersion $y \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ such that $y \cdot e_3 = v$. This is an important problem in plate theory and in our Γ-convergence setting, whose solution is crucial to the construction of recovery sequences. The question can be equivalently rephrased as: given an out-of-plane displacement v, is there an in-plane displacement u such that $(x' + u, v)$ is an isometric immersion of S? In [FJM06, Section 8], the authors show that a necessary and sufficient condition for the existence of u, or equivalently of y, is that $\det((\nabla')^2 v) = 0$. For the convenience of the reader, we recall here the aforementioned result.

Theorem 2.2.9. *Suppose that S is simply connected. Let $v \in W^{2,2}(S) \cap W^{1,\infty}(S)$ such that $\|\nabla' v\|_{L^\infty} < 1$. Then, there exists $u \in W^{2,2}(S;\mathbb{R}^2)$ such that*

$$y(x') := \begin{pmatrix} x' \\ 0 \end{pmatrix} + \begin{pmatrix} u(x') \\ v(x') \end{pmatrix} =: \begin{pmatrix} \phi(x') \\ v(x') \end{pmatrix}$$

is an isometric immersion if and only if $\det((\nabla')^2 v) = 0$. Moreover, if $\|\nabla' v\|_{L^\infty} \leq 1/2$ the function u can be chosen such that

$$\|(\nabla')^2 u\|_{L^2} \leq C \|\nabla' v\|_{L^\infty} \|(\nabla')^2 v\|_{L^2}, \tag{2.10}$$

$$\|u\|_{W^{2,2}} \leq C(\|\nabla' v\|_{L^\infty} \|(\nabla')^2 v\|_{L^2} + \|\nabla' v\|^2_{L^2}). \tag{2.11}$$

The condition $\det((\nabla')^2 v) = 0$ is a nonlinear elliptic PDE known as the Monge–Ampère equation. We define the set of its strong solutions as

$$\mathcal{A}_{\text{det}} := \Big\{ v \in W^{2,2}(S) \colon \det((\nabla')^2 v) = 0 \text{ a.e. in } S \Big\}.$$

The first result we prove is a slight generalization of Theorem 2.2.9, and it is concerned with the regularity of u.

Theorem 2.2.10. *Suppose that S is simply connected. Let $v \in W^{2,\infty}(S)$ and suppose that $\|\nabla' v\|_{L^\infty} < 1$. Then there is $u \in W^{2,\infty}(S;\mathbb{R}^2)$ such that the map*

$$y(x') = \begin{pmatrix} x' \\ 0 \end{pmatrix} + \begin{pmatrix} u(x') \\ v(x') \end{pmatrix}$$

is an isometric immersion if and only if $v \in \mathcal{A}_{\text{det}}$. Moreover, if $\|\nabla' v\|_{L^\infty} \leq 1/2$, the function u can be chosen such that

$$\|u\|_{W^{2,\infty}} \leq C(\|(\nabla')^2 v\|_{L^\infty} \|\nabla' v\|_{L^\infty} + \|\nabla' v\|^2_{L^\infty}). \tag{2.12}$$

Proof. The existence of $u \in W^{2,2}(S;\mathbb{R}^2)$ such that y is an isometric immersion is proved in Theorem 2.2.9. We are left to show that $u \in W^{2,\infty}(S;\mathbb{R}^2)$ and that (2.12) holds. In order to do so, we need to analyse the construction of u. We borrow the notation from the proof of Theorem 2.2.9 contained in [FJM06, Section 8]. Let

$$F := \sqrt{\mathrm{Id} - \nabla' v \otimes \nabla' v},$$

and

$$h_F := \frac{1}{\det(F)} F^T \operatorname{curl}(F).$$

Then, u is defined as $u(x') := \phi(x') - x'$, where $\phi \in W^{2,2}(S;\mathbb{R}^2)$ is such that $\nabla'\phi = e^{i\theta}F$ and $\theta \in W^{1,1}(S)$ has zero mean and satisfies $\nabla'\theta = h_F$. Here, $e^{i\theta}$ stands for the rotation matrix of angle θ

$$e^{i\theta} := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Note that

$$\det(F) = \sqrt{\det(\mathrm{Id} - \nabla' v \otimes \nabla' v)} = \sqrt{1 - |\nabla' v|^2} \geq \frac{1}{2}.$$

It is well-known that the matrix square root is differentiable and Lipschitz on the set of matrices whose determinant is positive and bounded away from 0. Thus, $F \in W^{1,\infty}(S;\mathbb{R}^{2\times 2})$ and

$$\begin{aligned} \|F\|_{L^\infty} &\leq C, \\ \|\nabla' F\|_{L^\infty} &\leq C\|(\nabla')^2 v\|_{L^\infty}\|\nabla' v\|_{L^\infty}. \end{aligned}$$

It follows that $h_F \in L^\infty(S;\mathbb{R}^2)$ and

$$\|\nabla'\theta\|_{L^\infty} = \|h_F\|_{L^\infty} \leq C\|(\nabla')^2 v\|_{L^\infty}\|\nabla' v\|_{L^\infty}.$$

Hence, we have

$$\begin{aligned} \|(\nabla')^2 u\|_{L^\infty} &= \|(\nabla')^2\phi\|_{L^\infty} \leq C(\|\nabla'\theta\|_{L^\infty}\|F\|_{L^\infty} + \|\nabla' F\|_{L^\infty}) \\ &\leq C\|(\nabla')^2 v\|_{L^\infty}\|\nabla' v\|_{L^\infty}, \\ \|\nabla' u\|_{L^\infty} &= \|\nabla'\phi - \mathrm{Id}\,\|_{L^\infty} \leq C\|F - \mathrm{Id}\,\|_{L^\infty} + \|e^{i\theta} - \mathrm{Id}\,\|_{L^\infty} \\ &\leq C\|F - \mathrm{Id}\,\|_{L^\infty} + \|\theta\|_{L^\infty} \leq C(\||\nabla' v|^2\|_{L^\infty} + \|\nabla'\theta\|_{L^\infty}) \\ &\leq C(\|(\nabla')^2 v\|_{L^\infty}\|\nabla' v\|_{L^\infty} + \|\nabla' v\|^2_{L^\infty}), \end{aligned}$$

where we have used the Poincaré–Wirtinger inequality on the term $\|\theta\|_{L^\infty}$ and a Taylor expansion of the matrix square root to treat the term $F - \mathrm{Id}$ (recall that the matrix square root has bounded derivative). Since u is defined up to translation, we conclude by applying the Poincaré–Wirtinger inequality. □

Since we are also interested in isometric immersions with a general constant metric $\mathcal{G} \in \mathbb{R}^{2\times 2}$, we would like to suitably extend the previous result. The datum v—which is the out-of-plane displacement in the Euclidean case—has to be changed to the displacement along a specific direction, related to the metric $\mathcal{G}$.

Theorem 2.2.11. *Let $U \in \mathbb{R}^{3\times 3}$ be a symmetric and positive definite matrix and let $\mathcal{G} := (U^2)'$. Suppose that S is simply connected. Let $v \in W^{2,2}(S)$ be such that*

$$|U^{-1}e_3| \|\nabla' v \mathcal{G}^{-1/2}\|_{L^\infty} < 1.$$

Then, there exists $\phi \in W^{2,2}(S;\mathbb{R}^2)$ such that the map

$$y(x') := vU^{-1}e_3 + U \begin{pmatrix} \phi \\ 0 \end{pmatrix} \tag{2.13}$$

belongs to $W^{2,2}_{\mathrm{iso},\mathcal{G}}(S;\mathbb{R}^3)$ if and only if $v \in \mathcal{A}_{\det}$. Moreover, if v satisfies the condition

$$|U^{-1}e_3| \|\nabla' v \mathcal{G}^{-1/2}\|_{L^\infty} \le \frac{1}{2},$$

then ϕ can be chosen such that $u := (U(\phi - x', 0))'$ satisfies the following estimates:

$$\|(\nabla')^2 u\|_{L^2} \le C\|\nabla' v\|_{L^\infty} \|(\nabla')^2 v\|_{L^2}, \tag{2.14}$$

$$\|u\|_{W^{2,2}} \le C\|\nabla' v\|_{L^\infty} \|(\nabla')^2 v\|_{L^2} + C\|\nabla' v\|^2_{L^2}. \tag{2.15}$$

Finally, if $v \in W^{2,\infty}(S)$, then $u \in W^{2,\infty}(S;\mathbb{R}^2)$ and the following inequality holds:

$$\|u\|_{W^{2,\infty}} \le C(\|(\nabla')^2 v\|_{L^\infty} \|\nabla' v\|_{L^\infty} + \|\nabla' v\|^2_{L^\infty}). \tag{2.16}$$

Proof. Observe that y of the form (2.13) satisfies $\nabla' y^T \nabla' y = \mathcal{G}$ if and only if

$$\nabla'\phi^T \mathcal{G} \nabla'\phi + |U^{-1}e_3|^2 \nabla' v \otimes \nabla' v = \mathcal{G}.$$

Defining $\tilde{\phi} := \mathcal{G}^{1/2}\phi$ and $\tilde{v} := |U^{-1}e_3| v$ the previous equation is equivalent to solve

$$(\nabla'\tilde{\phi}\mathcal{G}^{-1/2})^T(\nabla'\tilde{\phi}\mathcal{G}^{-1/2}) + (\nabla'\tilde{v}\mathcal{G}^{-1/2})^T \otimes (\nabla'\tilde{v}\mathcal{G}^{-1/2})^T = \mathrm{Id}_2$$

for the unknown $\tilde{\phi}$. Define now $\bar{v} \in W^{2,2}(\mathcal{G}^{1/2}S;\mathbb{R}^3)$ given by $\bar{v}(\mathcal{G}^{1/2}x') = \tilde{v}(x')$. Solving the above equation for $\tilde{\phi}$ is equivalent to solve

$$(\nabla'\bar{\phi})^T \nabla'\bar{\phi} + \nabla'\bar{v} \otimes \nabla'\bar{v} = \mathrm{Id}_2,$$

for $\bar{\phi} \in W^{1,2}(\mathcal{G}^{1/2}S;\mathbb{R}^2)$. Thus, we can conclude by applying Theorem 2.2.9. Note that the transformations $v \rightsquigarrow \tilde{v} \rightsquigarrow \bar{v}$ preserve the property of being a solution of the Monge–Ampère equation, namely

$$\det((\nabla')^2 v) = 0 \iff \det((\nabla')^2 \tilde{v}) = 0 \iff \det((\nabla')^2 \bar{v}) = 0.$$

Moreover, they preserve the boundedness of the gradient in the L^∞ norm. Finally, define $\bar{u} := \bar{\phi} - x'$. It is easy to show that

$$\begin{aligned}
&\|\nabla' u\|_{L^2} = C\|\nabla'\bar{u}\|_{L^2},\\
&\|u\|_{W^{1,2}} = C\|\bar{u}\|_{W^{1,2}},\\
&\|\nabla' v\|_{L^\infty} = C\|\nabla'\bar{v}\|_{L^\infty},\\
&\|(\nabla')^2 v\|_{L^2} = C\|(\nabla')^2\bar{v}\|_{L^2}.
\end{aligned}$$

Estimates (2.14)–(2.15) then follow from (2.10)–(2.11). We are left to prove (2.16). Note that it is sufficient to do the proof for $\bar{u}$, so we can suppose $U = \mathrm{Id}$. Then, the result follows from Theorem 2.2.10. $\square$

The latter result allows us to construct isometric immersions of S into $\mathbb{R}^3$ for the flat metric $(U^2)'$, given a suitable displacement along $U^{-1}e_3$ with respect to the reference deformation $x \mapsto Ux$. Note that $U^{-1}e_3$ is perpendicular to both Ue_1 and Ue_2.

When constructing recovery sequences, it is sometimes useful to approximate a Sobolev isometric immersion with a smooth one. We recall here a result proved by Hornung in [Hor11] giving a sufficient condition to have density of smooth isometric immersions in the space of Sobolev ones. We introduce a regularity condition for the boundary of S, that corresponds to condition $(*)$ in [Hor11].

$$\begin{gathered}\text{there is a closed subset } \Sigma \subset \partial S \text{ with } \mathcal{H}^1(\Sigma) = 0 \text{ such that}\\ \text{the outer unit normal } \vec{n} \text{ to } S \text{ exists and is continuous on } \partial S \backslash \Sigma.\end{gathered} \tag{2.17}$$

Theorem 2.2.12 (Hornung [Hor11])**.** *Suppose that S satisfies condition (2.17). Then the closure of the set*

$$W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3) \cap C^\infty(\bar{S};\mathbb{R}^3)$$

in $W^{2,2}(S;\mathbb{R}^3)$ is $W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$.

As a corollary, we state a density result for smooth solutions of the Monge–Ampère equation in the space of solutions with Sobolev regularity.

Corollary 2.2.13. *Suppose that S is simply connected and satisfies (2.17). Then the closure of*

$$\mathcal{A}_{\mathrm{det}} \cap C^\infty(\bar{S})$$

in $W^{2,2}(S)$ is $\mathcal{A}_{\mathrm{det}}$.

Proof. We need to show that for every $v \in \mathcal{A}_{\mathrm{det}}$ there is a sequence $(v_n) \subset \mathcal{A}_{\mathrm{det}} \cap C^\infty(\bar{S})$ such that $v_n \to v$ in $W^{2,2}(S)$. Suppose first that $v \in \mathcal{A}_{\mathrm{det}} \cap W^{1,\infty}(S)$. Take $\lambda \in \mathbb{R}$ such that $\|\nabla v\|_{L^\infty} < 1/\lambda$ and define $v^\lambda := \lambda v$. Clearly $\|\nabla v^\lambda\|_{L^\infty} < 1$. By Theorem 2.2.9 there is $\phi_\lambda \in W^{2,2}(S;\mathbb{R}^2)$ such that

$$y(x') := v^\lambda e_3 + \begin{pmatrix}\phi_\lambda \\ 0\end{pmatrix}$$

satisfies $\nabla' y^T \nabla' y = \mathrm{Id}$. By Theorem 2.2.12, there is a sequence of isometric immersions

$$(y_n) \subset W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3) \cap C^\infty(\overline{S};\mathbb{R}^3)$$

such that $y_n \to y$ in $W^{2,2}(S;\mathbb{R}^3)$. Defining $v_n := (1/\lambda) y_n \cdot e_3$ we have $v_n \to v$ in $W^{2,2}(S)$. Moreover, by Theorem 2.2.9, we deduce that $v_n \in \mathcal{A}_{\mathrm{det}} \cap C^\infty(\bar{S})$.

We move now to the general case of $v \in \mathcal{A}_{\mathrm{det}}$. By [FJM06, Theorem 10], there is a sequence $v_k \in \mathcal{A}_{\mathrm{det}} \cap W^{1,\infty}(S)$ such that $v_k \to v$ in $W^{2,2}$. Then, by a standard diagonal argument, we conclude. □

We present now a density results in L^2 for Sobolev solutions of the Monge–Ampère equation, which is of interest in its own right. The proof relies on the well-known Universal Approximation Theorem for Neural Networks (see [Cyb89; CD89]).

Theorem 2.2.14. *The set* span $\mathcal{A}_{\det}$ *is dense in* $L^2(S)$.

Proof. **Step 1**. Let $\sigma(x) := \dfrac{1}{1+e^{-x}}$. We show that σ is discriminatory, i.e., the only signed bounded regular Borel measure μ on $\bar{S}$ such that

$$\int_{\bar{S}} \sigma(y^T x + \theta)\, d\mu(x) = 0 \qquad \forall y \in \mathbb{R}^2, \forall \theta \in \mathbb{R} \tag{2.18}$$

is $\mu = 0$.

Let μ be such that (2.18) holds. We argue as in [Cyb89, Lemma 1]. Let $y \in \mathbb{R}^2$, $\lambda, \theta, k \in \mathbb{R}$, and define

$$\sigma_\lambda^k(x) := \sigma(\lambda(y^T x + \theta) + k).$$

Let

$$\phi^k(x) := \begin{cases} 1 & \text{if } y^T x + \theta > 0, \\ 0 & \text{if } y^T x + \theta < 0, \\ \sigma(k) & \text{if } y^T x + \theta = 0. \end{cases}$$

Clearly, $\sigma_\lambda^k \to \phi^k$ pointwise as $\lambda \to +\infty$. Moreover, $\|\sigma_\lambda^k\|_{C^0} \leq 1$ uniformly in λ. Hence, by (2.18) and dominated convergence

$$0 = \int_{\bar{S}} \sigma_\lambda^k \, d\mu \to \int_{\bar{S}} \phi^k \, d\mu = \sigma(k)\mu(\Pi_{y,\theta}) + \mu(H_{y,\theta}) \qquad \forall y \in \mathbb{R}^2, \forall \theta, k \in \mathbb{R}, \tag{2.19}$$

where

$$\Pi_{y,\theta} := \{x \in S \colon y^T x + \theta = 0\},$$
$$H_{y,\theta} := \{x \in S \colon y^T x + \theta > 0\}.$$

Passing to the limit as $k \to +\infty$ in (2.19), we deduce that

$$\mu(\Pi_{y,\theta}) + \mu(H_{y,\theta}) = 0 \qquad \forall y \in \mathbb{R}^2, \forall \theta \in \mathbb{R}.$$

Similarly, letting $k \to -\infty$ we get

$$\mu(H_{y,\theta}) = 0 \qquad \forall y \in \mathbb{R}^2, \forall \theta \in \mathbb{R}.$$

Fix $y \in \mathbb{R}^2$ and define

$$F_y(h) := \int_{\bar{S}} h(y^T x)\, d\mu(x),$$

for every bounded Borel function $h : \mathbb{R} \to \mathbb{R}$. Let $\theta \in \mathbb{R}$. Then,

$$F_y(\chi_{[-\theta,+\infty)}) = \mu(\Pi_{y,\theta}) + \mu(H_{y,\theta}) = 0,$$

where $\chi_{[-\theta,+\infty)}$ is the indicator function of $[-\theta, +\infty)$. Similarly, $F_y(\chi_{(-\theta,+\infty)}) = 0$. By the linearity of F_y, we deduce that F_y is zero on the indicator function of every interval. By approximation, $F_y(h) = 0$ for every continuous and bounded function $h : \mathbb{R} \to \mathbb{R}$ and for every $y \in \mathbb{R}^2$. In particular,

$$\begin{aligned} \widehat{\mu}(\xi) &= \int_{\bar{S}} e^{-i\xi^T x}\, d\mu(x) = \int_{\bar{S}} (\cos(\xi^T x) + i \sin(\xi^T x))\, d\mu(x) \\ &= F_\xi(\cos(x)) + i F_\xi(\sin(x)) = 0, \end{aligned}$$

where $\widehat{\mu}$ is the Fourier's transform of μ (see also Section 6.2 for a short introduction of the Fourier's Transform). Since $\widehat{\mu} = 0$, it follows that $\mu = 0$.

Step 2. Let

$$\Sigma := \left\{ x \mapsto \sigma(y^T x + \theta) \colon \theta \in \mathbb{R},\, y \in \mathbb{R}^2 \right\}.$$

We show that $\operatorname{span}\Sigma$ is dense in $C^0(\bar{S})$ with respect to the C^0 norm. Suppose by contradiction that $R = \overline{\operatorname{span}\Sigma} \subsetneq C^0(\bar{S})$. Then, by the Hahn–Banach Theorem, there is $L \in (C^0(\bar{S}))^*$ such that $L \neq 0$ and $L(R) = 0$. By the Riesz Representation Theorem, there is a signed bounded regular Borel measure $\mu \neq 0$ on $\bar{S}$ such that

$$L(h) = \int_{\bar{S}} h \, d\mu \qquad \forall\, h \in C^0(\bar{S}).$$

Since σ is discriminatory by Step 1, we have the desired contradiction.

Step 3. To conclude it is sufficient to show that $\Sigma \subset \mathcal{A}_{\det}$. Let $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^2$. We have

$$\nabla_x^2 \sigma(y^T x + \theta) = \sigma''(y^T x + \theta) \begin{pmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{pmatrix}.$$

Thus, $\det(\nabla_x^2 \sigma(y^T x + \theta)) = 0$, concluding the proof. $\square$

To conclude the section, we report a result proved in [Fre+16], regarding the construction of isometric immersions of narrow strips. More precisely, the following Theorem allows us—under suitable hypotheses—to extend a second fundamental form defined on the mid-line to a narrow strip, in such a way that it is still the second fundamental form of an isometric immersion. For this last part, $S := (0, L)$, so that $\Omega_h = (0, L) \times (-h/2, h/2)$.

Theorem 2.2.15. *Let $\rho > 0$ and let $p \in C^1([-\rho, L + \rho]; \mathbb{R}^2)$ be such that $|p| \equiv 1$ and $p \cdot e_1 \neq 0$ on $[-\rho, L + \rho]$. Then, there exists $\eta > 0$ and $O \subset \mathbb{R}^2$ a neighbourhood of $[0, L] \times \{0\}$ such that the map*

$$\Phi : [-\rho, L + \rho] \times (-\eta/2, \eta/2) \to \mathbb{R}^2, \qquad \Phi(x_1, x_2) := x_1 e_1 + x_2 p^\perp(x_1)$$

is a bi-Lipschitz homeomorphism onto O. In particular, there exists $\varepsilon > 0$, depending solely on p, such that $\Omega_\varepsilon \subset O$.

Consider now $y \in W^{2,2}(0, L; \mathbb{R}^3)$ and $d_2 \in W^{1,2}(0, L; \mathbb{R}^3)$ such that, defining $d_1 := \partial_1 y$, it holds:

(i) *$|d_1| = |d_2| = 1$ almost everywhere in $(0, L)$,*

(ii) *$d_1 \cdot d_2 = \partial_1 d_1 \cdot d_2 = 0$ almost everywhere in $(0, L)$.*

Assume the exists $\lambda \in L^2(0, L)$ such that $M := \lambda p \otimes p$ satisfies

$$M_{11} = \partial_1 d_1 \cdot d_3,$$
$$M_{12} = \partial_1 d_2 \cdot d_3,$$

where $d_3 := d_1 \wedge d_2$. For $(x_1, x_2) \in \Phi^{-1}(\Omega_\varepsilon)$, define u by the relation

$$u(\Phi(x_1, x_2)) := y(x_1) + x_2 \begin{pmatrix} d_1(x_1) & d_2(x_1) \end{pmatrix} p^\perp(x_1).$$

Then,

$$u \in W^{2,2}_{\mathrm{iso}}(\Omega_\varepsilon; \mathbb{R}^3) \cap W^{1,\infty}(\Omega_\varepsilon; \mathbb{R}^3)$$

and satisfies for almost every $x_1 \in (0, L)$

(a) $u(x_1, 0) = y(x_1)$,

(b) $\nabla u(\Phi(x_1, x_2)) = d_1(x_1) \otimes e_1 + d_2(x_1) \otimes e_2$,

(c) $\partial_{ij} u(x_1, 0) \cdot \nu(x_1, 0) = M_{ij}(x_1) = \lambda(x_1) p_i(x_1) p_j(x_1)$ *for* $i, j = 1, 2$, *where* $\nu := \partial_1 u \wedge \partial_2 u$.

For a proof, we refer to [Fre+16, Lemma 12 and Proposition 13].

2.2.3 Fine properties of optimal rotations

In this section, we recall some properties of optimal rotations (see Section 2.1 for their definition), and we further analyse their structure in our specific setting.

The first part is devoted to the projection operator as defined in Section 2.1, and its results are used in Chapter 3. We show that, under suitable hypotheses, P_h^j is well-defined, at least for $h \ll 1$, along sequences of rotations converging to an optimal one.

In the second part, we restrict our attention to the single-well case, and we investigate the consequences of a compatibility condition between f and $\mathcal{R}$ that is important in Chapter 4. In particular, in this case we give an explicit characterization of the tangent and normal space to $\mathcal{R}$.

Throughout this section, S represents the mid-plane of a plate, so that $S \subset \mathbb{R}^2$ is an open, bounded, and connected set with Lipschitz boundary.

For the first part of this section, let $(f_h) \subset L^2(S; \mathbb{R}^3)$ be a sequence of loads such that $h^{-p} f_h \to f$ in $L^2(S; \mathbb{R}^3)$ for some $p > 0$. Suppose that for every $h > 0$

$$\int_S f_h \, dx' = 0.$$

The first Lemma is an easy Γ-convergence result, whose proof is omitted. We recall that F_h and F are defined as in Section 2.1.

Lemma 2.2.16. *The sequence of functionals* $-h^{-p} F_h$ Γ*-converges to* $-F$. *In particular, given a sequence* $(R_h U_{k_h})$ *such that* $R_h U_{k_h} \in \mathcal{M}_h$ *for every* h, *up to a subsequence we have* $R_h U_{k_h} \to R U_j \in \mathcal{M}$.

Lemma 2.2.17. *Let* $j \in \{1, \ldots, l\}$. *Suppose that* $\dim \mathcal{R}_h^j \to \dim \mathcal{R}^j$. *Let* $(R_h) \subset \mathrm{SO}(3)$ *such that* $R_h \in \mathcal{R}_h^j$ *for every* h *and let* $(W_h) \subset \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ *be a sequence such that*

(a) $R_h W_h \in \mathrm{N}\mathcal{R}^j_{h\, R_h}$,

(b) $|W_h| = 1$ *for every* h.

Then, up to a subsequence, we have $R_h W_h \to RW$, *where* $R \in \mathcal{R}^j$, $W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$, $|W| = 1$, *and* $RW \in N\mathcal{R}^j_R$.

Proof. Up to subsequences, we have that $R_h \to R$ and $W_h \to W$ with $R \in \mathrm{SO}(3)$, $W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$, and $|W| = 1$. By Lemma 2.2.16 we have that $R \in \mathcal{R}^j$, thus we just need to prove that $RW \in N\mathcal{R}^j_R$. Let $m := \dim \mathcal{R}^j$. By hypothesis, $m = \dim \mathcal{R}_h^j$ for $h \ll 1$. Consider an orthonormal basis

$$\{R_h W_h^1, \ldots, R_h W_h^m\}$$

of the tangent space $T\mathcal{R}^j_{hR_h}$. Then, up to a subsequence, we have $R_hW^i_h \to RW^i$ for some $W^i \in \mathbb{R}^{3\times3}_{\text{skew}}$. Clearly the matrices $RW^1, \dots, RW^m$ are orthonormal. Moreover, since

$$0 = h^{-p}F_h(R_h(W^i_h)^2U_j) \to F(R(W^i)^2U_j) \qquad \forall\, i = 1, \dots, m$$

it follows that

$$\{RW^1U_j, \dots, RW^mU_j\}$$

is an orthonormal basis of $T\mathcal{R}^j_R$.

Consider now a matrix $M \in \mathbb{R}^{3\times3}_{\text{skew}}$ such that $RM \in T\mathcal{R}^j_R$. Then, we can write $RM = \sum_{i=1}^m \lambda_i RW^i$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Define $M_h := \sum_{i=1}^m \lambda_i W^i_h$. By construction, we have $R_hM_h \to RM$. Moreover, R_hM_h is tangent to $\mathcal{R}^j_h$ at the point R_h, so that

$$0 = R_hW_h : R_hM_h \to RW : RM.$$

Since M is arbitrary, this concludes the proof. □

Remark 2.2.18. Lemma 2.2.17 proves also that a sequence of tangent matrices to $\mathcal{R}^j_h$ at a point R_h converges to a tangent matrix to $\mathcal{R}^j$ at the point R, where R is the limit of R_h.

Proposition 2.2.19. *Let $j \in \{1, \dots, l\}$. Suppose that* $\dim \mathcal{R}^j_h \to \dim \mathcal{R}^j$. *Let*

$$\widetilde{\mathcal{R}}^j := \left\{R \in \mathcal{R}^j : \exists\, (R_h) \subset \mathrm{SO}(3) \text{ s.t. } R_h \in \mathcal{R}^j_h \text{ for every } h > 0 \text{ and } R_h \to R\right\}.$$

Then $\widetilde{\mathcal{R}}^j = \mathcal{R}^j$.

Proof. We show that $\widetilde{\mathcal{R}}^j$ is the image of $T\mathcal{R}^j_R$ through the map

$$T\mathcal{R}^j_R \to \mathrm{SO}(3), \qquad RW \mapsto Re^W,$$

in a neighbourhood of R. In particular, this proves that $\widetilde{\mathcal{R}}^j$ is an embedded submanifold of $\mathcal{R}^j$ and that the tangent spaces coincide, concluding the proof.

Let $R \in \widetilde{\mathcal{R}}^j$. There exists a sequence $(R_h) \subset \mathrm{SO}(3)$ such that $R_h \in \mathcal{R}^j_h$ for every h and $R_h \to R$. For $h \ll 1$, take an orthonormal basis $\{R_hW^1_h, \dots, R_hW^m_h\}$ of $T\mathcal{R}^j_{hR_h}$, where $m := \dim \mathcal{R}^j_h = \dim \mathcal{R}^j$. Then $R_hW^i_h \to RW^i$ and since

$$0 = \frac{1}{h^p}F_h(R(W^i_h)^2U_j) \to F(R(W^i)^2U_j),$$

the set $\{RW^1, \dots, RW^m\}$ is an orthonormal basis of $T\mathcal{R}^j_R$. Now pick $W \in T\mathcal{R}^j_R$. By the convergence of the basis we can construct a sequence $(W_h) \subset \mathbb{R}^{3\times3}_{\text{skew}}$ such that $W_h \to W$ and $R_hW_h \in T\mathcal{R}^j_h$ for every $h \ll 1$. By (2.4), we have $R_he^{W_h} \in \mathcal{R}^j_h$. Thus, passing to the limit, we get by Lemma 2.2.16 that $Re^W \in \mathcal{R}^j$, that is, by definition $Re^W \in \widetilde{\mathcal{R}}^j$. □

The above results grant the well-posedness of the projection.

Proposition 2.2.20. *Let $(R_h) \subset \mathrm{SO}(3)$ be a sequence such that $R_h \to R$ and $R \in \mathcal{R}^j$ for some $j \in \{1, \dots, l\}$. Suppose that $\dim \mathcal{R}^j_h \to \dim \mathcal{R}^j$. Then $P^j_h(R_h)$ is well-defined for $h \ll 1$.*

Proof. It is sufficient to prove that $\mathrm{dist}_{\mathrm{SO}(3)}(R_h, \mathcal{R}^j_h) \to 0$. By Proposition 2.2.19, there exists a sequence of rotations $\tilde{R}_h$ such that $\tilde{R}_h \in \mathcal{R}^j_h$ for every h and $\tilde{R}_h \to R$. Since

$$\mathrm{dist}_{\mathrm{SO}(3)}(Q', Q) = |Q' - Q| + O(|Q' - Q|^2)$$

for every $Q, Q' \in \mathrm{SO}(3)$, we have

$$\mathrm{dist}_{\mathrm{SO}(3)}(R_h, \mathcal{R}^j_h) \leq \mathrm{dist}_{SO(3)}(R_h, \tilde{R}_h) = |R_h - \tilde{R}_h| + O(|R_h - \tilde{R}_h|^2) \to 0.$$

$\square$

We move now to the second part of the section, and we restrict our attention to the single-well case (i.e., $l := 1$, $U_1 := \mathrm{Id}$ so that $K_{\mathcal{W}} = \mathrm{SO}(3)$). We use the reduced notation $\mathcal{R}$ to denote both $\mathcal{M}$ and $\mathcal{R}^1$.

Consider a non-zero force $f \in L^2(S; \mathbb{R}^3)$ and suppose that

$$\int_S f \, dx' = 0.$$

We start with some result regarding the dimension of $\mathcal{R}$. Firstly, we recall the characterization contained in [MM21, Proposition 6.2].

Proposition 2.2.21. *Let $L : \mathrm{SO}(3) \to \mathbb{R}$ be a linear map and let $\mathcal{R}_L$ be the set of its maximizers. Suppose that $\mathrm{Id} \in \mathcal{R}_L$. With a small abuse of notation, let L be the 3×3 matrix representing the linear function L. Either we have $\mathcal{R}_L = \{\mathrm{Id}\}$ or*

(i) *$\mathcal{R}_L = \mathrm{SO}(3)$, if and only if $L = 0$,*

(ii) *$\mathcal{R}_L$ is isometric to the real projective plane $\mathbb{P}_2(\mathbb{R})$, if and only if the eigenvalues of L are a, a, $-a$ for some $a > 0$,*

(iii) *$\mathcal{R}_L$ is a single closed geodesic, if and only if the eigenvalues of L are b, a, $-a$ for some $b > a \geq 0$.*

We show that, in our setting, case (ii) is not admissible.

Lemma 2.2.22. *The dimension of $\mathcal{R}$ is not 2.*

Proof. Let $\bar{R} \in \mathcal{R}$. Define

$$\widetilde{F}(A) := \int_\Omega f \cdot \bar{R} A \begin{pmatrix} x' \\ 0 \end{pmatrix} dx = \int_\Omega \bar{R}^T f \cdot A \begin{pmatrix} x' \\ 0 \end{pmatrix} dx.$$

Similarly, define

$$\widetilde{\mathcal{R}} := \operatorname*{argmax}_{R \in \mathrm{SO}(3)} \widetilde{F}(R).$$

Note that $\mathcal{R} = \bar{R} \cdot \widetilde{\mathcal{R}}$ so it is enough to prove that $\dim \widetilde{\mathcal{R}} \neq 2$. Clearly $\mathrm{Id} \in \widetilde{\mathcal{R}}$, so we can use the classification of Proposition 2.2.21. Since $\widetilde{F}$ and F are linear on the space of 3×3 matrices, we can represent them by 3×3 matrices, that we still denote, with a slight abuse of notation, by $\widetilde{F}$ and F. By Proposition 2.2.21, $\mathcal{R}$ is two-dimensional when the eigenvalues of $\widetilde{F}$ are of the form $a, a, -a$ for some $a > 0$. Note first of all that

$$\widetilde{F} : A = F : \bar{R}A \qquad \forall\, A \in \mathbb{R}^{3\times 3},$$

so that $\widetilde{F} = \bar{R}^T F$. Moreover,

$$F_{i3} = F : E^{i3} = F(E^{i3}) = 0, \qquad i = 1, 2, 3,$$

where E^{ij} is the matrix such that $E^{ij}_{km} := \delta_{ki}\delta_{mj}$ and δ_{ij} is the usual Kronecker's symbol. It follows that $\det(\widetilde{F}) = \det(F) = 0$ and 0 is an eigenvalue of $\widetilde{F}$, concluding the proof. □

In this part of the section, we are mainly interested in the consequences of the compatibility condition

$$R^T f \cdot e_3 = 0 \qquad \forall\, R \in \mathcal{R}. \tag{C}$$

Condition (C) plays an important role in Chapter 4.

Remark 2.2.23. If (C) is in force, then we also have $\dim \mathcal{R} \neq 3$, thus $\mathcal{R}$ is either a singleton or a closed geodesic. However, as showed in [MM21], we can have non-zero forces for which $\mathcal{R} = \mathrm{SO}(3)$. As an example consider $f := (1 - 3/2|x|)e_1$ acting on $S := B_1$. Then

$$F(R) = \int_{B_1} f(x) \cdot R \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' = \int_0^1 \int_0^{2\pi} r \left(1 - \frac{3}{2} r\right) e_1 \cdot R \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} d\theta\, dr = 0.$$

In particular $\mathcal{R} = \mathrm{SO}(3)$. In this case, (C) does not hold.

The set of rotations is not linear. However, the 2-dimensional structure of the integral that defines F gives us the freedom to perform some change of sign to the columns of a rotation while keeping the sign of its determinant. A few simple results follow from this observation.

Lemma 2.2.24. *If (C) holds, then*

$$\max_{R \in \mathrm{SO}(3)} F(R) > 0.$$

Otherwise,

$$\max_{R \in \mathrm{SO}(3)} F(R) \geq 0.$$

Proof. Assume (C) and suppose by contradiction that $F(R) \leq 0$ for any rotation $R \in \mathrm{SO}(3)$. By (C) we have $\mathcal{R} \neq \mathrm{SO}(3)$, hence the map F can not vanish on the whole $\mathrm{SO}(3)$. Thus, there is a rotation R such that $F(R) < 0$. Now consider the matrix

$$\hat{R} := \begin{pmatrix} -R^1 & -R^2 & R^3 \end{pmatrix}.$$

Note that $\hat{R} \in \mathrm{SO}(3)$ and $F(\hat{R}) = -F(R) > 0$. This gives the desired contradiction. The same argument applies to the second part of the statement. □

Differentiating the map $t \mapsto F(Re^{tW})$ and evaluating it at $t = 0$ we obtain

$$F(RW) = 0,\ F(RW^2) \leq 0 \qquad \forall\, R \in \mathcal{R},\ \forall\, W \in \mathbb{R}^{3\times 3}_{\text{skew}}. \tag{2.20}$$

Consider now $R \in \mathcal{R}$ and the skew-symmetric matrix

$$W := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $F(RW) = 0$, we get

$$\int_S f \cdot R \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} dx' = \int_S f \cdot R \begin{pmatrix} 0 \\ x_1 \\ 0 \end{pmatrix} dx'.$$

For a given $R \in \mathcal{R}$ we then define

$$a(R) := \int_S f \cdot R \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} dx', \tag{2.21}$$

$$b(R) := \int_S f \cdot R \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} dx', \tag{2.22}$$

$$c(R) := \int_S f \cdot R \begin{pmatrix} 0 \\ x_1 \\ 0 \end{pmatrix} dx' = \int_S f \cdot R \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} dx. \tag{2.23}$$

Note that by Lemma 2.2.24 we have that $a(R) + b(R) = F(R) \geq 0$. Moreover, $a(R)$ and $b(R)$ can not be negative, as proved in the following lemma. In particular, when (C) holds, $a(R)$ and $b(R)$ cannot be both zero by Lemma 2.2.24.

Lemma 2.2.25. *It holds that* $a(R), b(R) \geq 0$ *for any* $R \in \mathcal{R}$.

Proof. Suppose by contradiction that $a(R) < 0$ for some $R \in \mathcal{R}$. By Lemma 2.2.24 we have $b(R) = F(R) - a(R) \geq 0$. Consider the rotation

$$\hat{R} := \begin{pmatrix} -R^1 & R^2 & -R^3 \end{pmatrix} \in \mathrm{SO}(3).$$

Then $F(R) \geq F(\hat{R}) = -a(R) + b(R) > a(R) + b(R) = F(R)$, which gives a contradiction. A similar proof can be done for $b(R)$. □

We can now give an explicit characterization of the tangent space $T\mathcal{R}_R$ in terms of the quantities $a(R), b(R)$ and $c(R)$.

Proposition 2.2.26. *Assume* (C) *and suppose that* $\dim \mathcal{R} = 1$. *Let* $R \in \mathcal{R}$. *Then*

$$a(R)b(R) - c^2(R) = 0.$$

Moreover,

$$T\mathcal{R}_R = \left\{ W \in \mathbb{R}^{3\times 3}_{\text{skew}} : W_{12} = 0,\ W_{13} = -\frac{c(R)}{a(R)} W_{23} \right\} \qquad \text{if } a(R) \neq 0,$$

$$T\mathcal{R}_R = \left\{ W \in \mathbb{R}^{3\times 3}_{\text{skew}} : W_{12} = 0,\ W_{23} = -\frac{c(R)}{b(R)} W_{13} \right\} \qquad \text{if } b(R) \neq 0.$$

Proof. By definition, the tangent space to $\mathcal{R}$ at R is the set of zeros of the map $W \in \mathbb{R}^{3\times 3}_{\text{skew}} \mapsto F(RW^2)$. For a general skew-symmetric matrix W, we have

$$(W^2)' = -\begin{pmatrix} W_{12}^2 + W_{13}^2 & W_{13}W_{23} \\ W_{13}W_{23} & W_{12}^2 + W_{23}^2 \end{pmatrix}.$$

Hence, by (C) we have

$$F(RW^2) = -(W_{12}^2 + W_{13}^2)a(R) - 2W_{13}W_{23}c(R) - (W_{12}^2 + W_{23}^2)b(R)$$

This expression can be considered as a quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ computed at the vector (W_{12}, W_{13}, W_{23}). We can identify q with a symmetric matrix and study its sign. We have

$$q = -\begin{pmatrix} a(R)+b(R) & 0 & 0 \\ 0 & a(R) & c(R) \\ 0 & c(R) & b(R) \end{pmatrix},$$

so by Lemma 2.2.24–2.2.25 the sign of q depends solely on the minor $a(R)b(R) - c^2(R)$. If $a(R)b(R) - c^2(R) > 0$, the only zero of q is at 0, contradicting the hypothesis on the dimension of $\mathcal{R}$. If $a(R)b(R) - c^2(R) < 0$, the set of zeros of q contains two lines that span a subset of dimension 2 in $\mathbb{R}^3$, contradicting again the assumption $\dim \mathcal{R} = 1$. Therefore, it must hold that $a(R)b(R) - c^2(R) = 0$. In this case, we have

$$q(W) = -W_{12}^2 F(R) - \left(W_{13}\sqrt{a(R)} + W_{23}\frac{c(R)}{\sqrt{a(R)}} \right)^2 \qquad \text{if } a(R) \neq 0,$$

$$q(W) = -W_{12}^2 F(R) - \left(W_{23}\sqrt{b(R)} + W_{13}\frac{c(R)}{\sqrt{b(R)}} \right)^2 \qquad \text{if } b(R) \neq 0,$$

concluding the characterization of the tangent space by Lemma 2.2.24. □

Corollary 2.2.27. *Assume (C), suppose that* $\dim \mathcal{R} = 1$*, and let* $R \in \mathcal{R}$*. Then*

$$\mathrm{N}\mathcal{R}_R = \left\{ W \in \mathbb{R}^{3\times 3}_{\text{skew}} : W_{23} = \frac{c(R)}{a(R)} W_{13} \right\} \qquad \text{if } a(R) \neq 0,$$

$$\mathrm{N}\mathcal{R}_R = \left\{ W \in \mathbb{R}^{3\times 3}_{\text{skew}} : W_{13} = \frac{c(R)}{b(R)} W_{23} \right\} \qquad \text{if } b(R) \neq 0.$$

3

Γ-convergence of a singularly perturbed multi-well energies

3.1 Assumptions and main results

In this chapter, we assume $S \subset \mathbb{R}^2$ to be an open, bounded, and connected set with Lipschitz boundary, representing the mid-plane of a plate. The elastic energy density $\mathcal{W}$ is defined on $\mathbb{R}^{3\times 3}$. Moreover, we assume that

$$\mathcal{W}(M) \geq C f_q(\mathrm{dist}(M,K)), \qquad M \in \mathbb{R}^{3\times 3}, \tag{3.1}$$

where $f_q := t^2 \wedge t^q$ and $q \in [0,2]$. This implies in particular the growth condition (2.8). Note that far from K (see Section 2.1 for the definition of K) the energy density may even have sublinear growth. We denote with the greek letter α a scaling exponent in $[2,+\infty)$, and we set $\gamma := \alpha/2$. We choose $p > 1$ and $\eta\colon (0,+\infty) \to (0,+\infty)$ such that for some constant $C > 0$:

(P1) $\eta(h) \geq Ch^{\frac{\alpha}{3}}$ for every $h > 0$,

(P2) $\eta(h)h^{\gamma(1-\frac{2}{p})-1} \to 0$ as $h \to 0$,

(P3) $p > 6/5$, if $q < 2$.

Conditions (P1) and (P3) ensure that the penalty term is strong enough to provide suitable compactness estimates (see Proposition 3.2.1) whereas condition (P2) guarantees that the penalty term is negligible at the limit. Note that (P1)–(P3) are compatible, since for every $\alpha \geq 2$ we have $1 - \gamma(1 - p/2) < \alpha/3$ for p large enough.

The symbol $\nabla_h^2 y$ denotes the rescaled Hessian of y, that is simply $\nabla_h(\nabla_h y)$. We set

$$W^h_{2,p}(\Omega;\mathbb{R}^3) := \Big\{ y \in W^{1,2}(\Omega;\mathbb{R}^3) \colon \nabla_h^2 y \in L^p(\Omega;\mathbb{R}^{3\times 3\times 3}) \Big\},$$

and we define the α-rescaled energy $E_h^\alpha \colon W^{1,2}(\Omega;\mathbb{R}^3) \to \mathbb{R}$ as follows:

$$E_h^\alpha(y) := \begin{cases} \dfrac{1}{h^\alpha}\displaystyle\int_\Omega \mathcal{W}(\nabla_h y)\,dx + \dfrac{\eta^p(h)}{h^\alpha}\displaystyle\int_\Omega |\nabla_h^2 y|^p\,dx & \text{if } y \in W^h_{2,p}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

The main objective of this chapter is to extend the rigorous derivation of the hierarchy of plate models obtained by Friesecke, James, and Müller in [FJM02; FJM06] for a single-well energy to the perturbed multi-well case. In order to do so, we need to introduce the limiting models centred at a well K_j.

The Kirchhoff's regime $\alpha = 2$

For $\alpha = 2$, we prove the Γ-convergence of E_h^α to the Kirchhoff's functional

$$E_j^{\mathrm{K}}(y) := \begin{cases} \dfrac{1}{24}\displaystyle\int_S \bar{Q}_j(\nabla y^T \nabla \nu)\,dx & y \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.2}$$

where $\mathcal{G}_j := (U_j^2)'$ and ν is the unique vector such that $\begin{pmatrix} \nabla' y & \nu \end{pmatrix} U_j^{-1} \in \mathrm{SO}(3)$ a.e. in S (whose existence is granted by Lemma 3.2.8). The Γ-convergence result is stated in the following theorem.

Theorem 3.1.1 (Γ-convergence ($\alpha = 2$))**.** *Suppose that S satisfies (2.17).*

(i) *For any sequence $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ that satisfies $E_h^2(y_h) \le C$ for every $h > 0$ there exist $y \in W^{2,2}(S;\mathbb{R}^3)$ and $j \in \{1,\dots,l\}$ such that:*

  (a) *$\nabla' y^T \nabla' y = (U_j^2)'$ or equivalently $y \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$,*

  (b) *up to a nonrelabelled subsequence, $\nabla_h y_h \to \begin{pmatrix} \nabla' y & \nu \end{pmatrix}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$, where ν is the unique vector such that $\begin{pmatrix} \nabla' y & \nu \end{pmatrix} U_j^{-1} \in \mathrm{SO}(3)$ a.e.*

(ii) *For any sequence $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ as in (i) it holds*

$$\liminf_{h\to 0} E_h^2(y_h) \ge E_j^{\mathrm{K}}(y).$$

(iii) *For any $y \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$, there exists a sequence $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that (b) holds true and*

$$\lim_{h\to 0} E_h^2(y_h) = E_j^{\mathrm{K}}(y).$$

The Von Kármán's regime $\alpha > 2$

When $\alpha > 2$, the limiting models are expressed in terms of u and v, the limits of the rescaled in-plane and out-of-plane averaged displacements around a well K_j, called u_h and v_h, respectively. They are defined as follows:

$$u_h\colon S \to \mathbb{R}^2 \qquad x' \mapsto \min\left\{h^{-\gamma}, h^{2-2\gamma}\right\} \begin{pmatrix} \omega_h(x') \cdot U_j e_1 \\ \omega_h(x') \cdot U_j e_2 \end{pmatrix}, \tag{3.3}$$

$$v_h\colon S \to \mathbb{R} \qquad x' \mapsto h^{1-\gamma}\omega_h(x') \cdot U_j e_3, \tag{3.4}$$

where we recall that $\gamma := \alpha/2$ and

$$\omega_h\colon S \to \mathbb{R}^3, \qquad x' \mapsto \int_I \left(y_h - U_j \begin{pmatrix} x' \\ hx_3 \end{pmatrix} \right) dx_3.$$

Observe that u_h and v_h are the components of the averaged displacement ω_h in the basis given by $\{U_j^{-i} e_i\colon i = 1,2,3\}$. This may not look as a natural choice, since a basis of tangent vectors to the embedded midplane $U_j(S \times \{0\})$ is given by $\{U_j e_1, U_j e_2\}$ and the normal direction is given by $U_j^{-1} e_3$. However, since $\{U_j^{-1} e_i : i = 1,2,3\}$ is the dual basis of $\{U_j e_i : i = 1,2,3\}$, this alternative simplifies both the statement and the computations, and gives a completely equivalent result (see Remark 3.2.12).

When $2 < \alpha < 4$, we retrieve the Γ-convergence to the constrained Von Kármán's functional, namely,

$$E_j^{\mathrm{CVK}}(v) := \begin{cases} \dfrac{1}{24} \displaystyle\int_S \bar{Q}_j((\nabla')^2 v)\, dx & \text{if } v \in \mathcal{A}^{\mathrm{lin}}_{\mathrm{iso},j}, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{A}^{\mathrm{lin}}_{\mathrm{iso},j} := \Big\{ v \in W^{2,2}(S) : \exists\, u \in W^{1,2}(S;\mathbb{R}^2) \\ \text{s.t. } \nabla' u^T + \nabla' u + |U_j^{-1} e_3|^2 \nabla' v \otimes \nabla' v = 0 \Big\}.$$

As in the single-well case, the constraint

$$\nabla' u^T + \nabla' u + |U_j^{-1} e_3|^2 \nabla' v \otimes \nabla' v = 0 \tag{3.5}$$

means that u and v satisfy a matching isometry condition up to the second order; however, the metric now depends on the well, and it is not necessarily the Euclidean one. More precisely, if one defines for $\varepsilon > 0$

$$y_\varepsilon := U_j \begin{pmatrix} x' \\ 0 \end{pmatrix} + \varepsilon U_j^{-1} e_3 v + \varepsilon^2 U_j^{-1} \begin{pmatrix} u \\ 0 \end{pmatrix},$$

then (3.5) is equivalent to $\nabla' y_\varepsilon^T \nabla' y_\varepsilon = (U_j^2)' + O(\varepsilon^3)$. In the literature, y_ε is called a geometrically linearized isometry. If S is simply connected, for a given v there exists u solving (3.5) if and only if $v \in \mathcal{A}_{\mathrm{det}}$, as we show in the next proposition.

Proposition 3.1.2. *Assume that S is simply connected. Let $v \in W^{2,2}(S)$. There exists $u \in W^{1,2}(S;\mathbb{R}^2)$ solving equation (3.5) if and only if $v \in \mathcal{A}_{\mathrm{det}}$.*

Proof. We follow [FJM06, Proposition 9]. Suppose first that $U_j := \mathrm{Id}$ and define

$$g := -\frac{1}{2}\nabla' v \otimes \nabla' v.$$

With some simple computations it is possible to prove that the relation

$$\partial_{22}g_{11} + \partial_{11}g_{22} - 2\partial_{12}g_{12} = \det((\nabla')^2 v)$$

holds in the sense of distributions. If we show that g is the symmetric gradient of a $W^{1,2}$ map if and only if

$$\partial_{22}g_{11} + \partial_{11}g_{22} - 2\partial_{12}g_{12} = 0, \tag{3.6}$$

the proof is concluded. Suppose first that $g = \mathrm{sym}(\nabla' u)$ for some $u \in W^{1,2}(S;\mathbb{R}^2)$. Then, in the sense of distributions,

$$\partial_{22}g_{11} + \partial_{11}g_{22} - 2\partial_{12}g_{12} = \partial_{221}u_1 + \partial_{112}u_2 - \partial_{12}(\partial_1 u_2 + \partial_2 u_1) = 0.$$

Assume now that g satisfies (3.6). We look for $u \in W^{1,2}(S;\mathbb{R}^2)$ such that

$$\begin{cases} \partial_1 u_1 = g_{11}, \\ \partial_2 u_2 = g_{22}, \\ \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) = g_{12} = g_{21}. \end{cases} \tag{3.7}$$

Note that the last equation can be equivalently rewritten as

$$\partial_1 u_2 = g_{21} + \frac{1}{2}\,\mathrm{curl}(u),$$
$$\partial_2 u_1 = g_{12} - \frac{1}{2}\,\mathrm{curl}(u).$$

Thus, solving (3.7) is equivalent to solve

$$\begin{cases} \nabla u = \begin{pmatrix} g_{11} & g_{12} - f \\ g_{21} + f & g_{22} \end{pmatrix}, \\ f = \frac{1}{2}\,\mathrm{curl}(u), \end{cases} \tag{3.8}$$

for $u \in W^{1,2}(S;\mathbb{R}^2)$ and $f \in L^2(S)$. We show now that the two equation can be uncoupled. Taking the distributional gradient of the last equation, we get

$$\nabla f = \frac{1}{2}\begin{pmatrix} \partial_{11}u_2 - \partial_{12}u_1 \\ \partial_{21}u_2 - \partial_{22}u_1 \end{pmatrix} = \begin{pmatrix} \partial_1 g_{12} - \partial_2 g_{11} \\ \partial_1 g_{22} - \partial_2 g_{12} \end{pmatrix}. \tag{3.9}$$

Recall that a vector-valued (or matrix-valued) distribution on a simply connected set is a distributional gradient if and only if its curl is zero (see also Section 6.5). Since

$$\mathrm{curl}\begin{pmatrix} \partial_1 g_{12} - \partial_2 g_{11} \\ \partial_1 g_{22} - \partial_2 g_{12} \end{pmatrix} = \partial_{22}g_{11} + \partial_{11}g_{22} - 2\partial_{12}g_{12} = 0,$$

equation (3.9) can be solved for $f \in L^2(S)$. Lastly, since

$$\operatorname{curl}\begin{pmatrix} g_{11} & g_{12} - f \\ g_{21} + f & g_{22} \end{pmatrix} = 0,$$

for the same reason the first equation in (3.8) can be solved for $u \in W^{1,2}(S;\mathbb{R}^2)$, concluding the proof. If $U_j \neq \mathrm{Id}$, it is sufficient to note that (3.5) can be rewritten as

$$\frac{1}{|U_j^{-1}e_3|^2}(\nabla' u^T + \nabla' u) + \nabla' v \otimes \nabla' v = \nabla' \tilde{u}^T + \nabla' \tilde{u} + \nabla' v \otimes \nabla' v = 0,$$

where $\tilde{u} := |U_j^{-1}e_3|^{-2}u$. □

The case $\alpha = 4$ corresponds to the Von Kármán's model, that is,

$$\begin{aligned} E_j^{\mathrm{VK}}(u,v) := & \frac{1}{24}\int_S \bar{Q}_j((\nabla')^2 v)\, dx \\ & + \frac{1}{8}\int_S \bar{Q}_j(\nabla' u^T + \nabla' u + |U_j^{-1}e_3|^2 \nabla' v \otimes \nabla' v)\, dx, \end{aligned}$$

for $v \in W^{2,2}(S)$ and $u \in W^{1,2}(S;\mathbb{R}^2)$.

Lastly, if $\alpha > 4$, we prove the Γ-convergence to the linearized Von Kármán's model

$$E_j^{\mathrm{LVK}}(u,v) := \frac{1}{24}\int_S \bar{Q}_j((\nabla')^2 v)\, dx + \frac{1}{8}\int_S \bar{Q}_j(\nabla' u^T + \nabla' u)\, dx,$$

for $v \in W^{2,2}(S)$ and $u \in W^{1,2}(S;\mathbb{R}^2)$. The following theorem summarizes these Γ-convergence results.

Theorem 3.1.3 (Γ-convergence ($\alpha > 2$)). *Suppose $\alpha > 2$.*

(i) *For any sequence $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ that satisfies $E_h^\alpha(y_h) \leq C$ for every $h > 0$ there exist an index $j \in \{1,\dots,l\}$, two sequences $(\bar{R}_h) \subset \mathrm{SO}(3)$, $(c_h) \subset \mathbb{R}$, and two maps $v \in W^{2,2}(S), u \in W^{1,2}(S;\mathbb{R}^2)$ such that, up to a subsequence, the following convergences hold:*

   (a) $u_h \rightharpoonup u$ *in* $W^{1,2}(S;\mathbb{R}^2)$,

   (b) $v_h \to v$ *in* $W^{1,2}(S)$,

   where u_h and v_h are, respectively, the in-plane and out-of-plane displacements around the well K_j defined as in (3.3)–(3.4) for the roto-translated deformation

$$\tilde{y}_h := \bar{R}_h^T y_h + c_h.$$

   Moreover, if $2 < \alpha < 4$, then

$$\nabla' u^T + \nabla' u + |U_j^{-1}e_3|^2 \nabla' v \otimes \nabla' v = 0.$$

(ii) *For any sequence $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ as in (i)*

(a) *if* $2 < \alpha < 4$, *then* $\liminf_{h\to 0} E_h^\alpha(y_h) \geq E_j^{\mathrm{CVK}}(v)$,

(b) *if* $\alpha = 4$, *then* $\liminf_{h\to 0} E_h^\alpha(y_h) \geq E_j^{\mathrm{VK}}(u, v)$,

(c) *if* $\alpha > 4$, *then* $\liminf_{h\to 0} E_h^\alpha(y_h) \geq E_j^{\mathrm{LVK}}(u, v)$.

(iii) *Suppose that* $2 < \alpha < 4$ *and that* S *is simply connected and satisfies* (2.17)*. For any choice of* $j \in \{1, \dots, l\}$ *and* $v \in \mathcal{A}^{\mathrm{lin}}_{\mathrm{iso},j}$, *there exists a sequence* $(y_h) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ *such that*

(a) $v_h \to v$ *in* $W^{1,2}(S)$, *where* v_h *is defined as in* (3.4)*,*

(b) $\lim_{h\to 0} E_h^\alpha(y_h) = E_j^{\mathrm{CVK}}(v)$.

(iv) *For any choice of* $j \in \{1, \dots, l\}$, $u \in W^{1,2}(S; \mathbb{R}^2)$, *and* $v \in W^{2,2}(S)$ *there exists a sequence* $(y_h) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ *such that, defining* u_h, v_h *as in* (3.3)*–*(3.4)*,*

(a) $v_h \to v$ *in* $W^{1,2}(S)$,

(b) $u_h \rightharpoonup u$ *in* $W^{1,2}(S; \mathbb{R}^2)$,

(c) $\lim_{h\to 0} E_h^\alpha(y_h) = E_j^{\mathrm{VK}}(u, v)$ *if* $\alpha = 4$,

(d) $\lim_{h\to 0} E_h^\alpha(y_h) = E_j^{\mathrm{LVK}}(u, v)$ *if* $\alpha > 4$.

Remark 3.1.4. In the proof of Theorem 3.1.3–(iii) we cannot use the truncation argument of [FJM02; FJM06]. Indeed, the penalty term in the energy requires higher regularity. To overcome this issue we suppose that S satisfies (2.17), so that we can apply the density result by Hornung recalled in Theorem 2.2.12.

Remark 3.1.5. In the single-well case, that is $l := 1$ and $U_1 := \mathrm{Id}$, Theorems 3.1.1 and 3.1.3 hold also for $\eta(h) \equiv 0$, namely, without penalty term. Indeed, this is precisely the setting of [FJM02; FJM06]. Moreover, if instead of rescaling by h^α we do it by a generic infinitesimal sequence $D_h \to 0$, we obtain the same hierarchy of Γ-limits, depending on the asymptotic behaviour of D_h/h^2 and D_h/h^4.

Convergence of minimizers in the presence of dead loads

External forces can be included in the previous analysis. For this part, we assume $q > 1$, excluding sublinear and linear growth of $\mathcal{W}$ at infinity, and we study the convergence of minimizers of the rescaled total energy

$$J_h^\alpha \colon W^{1,2}(\Omega; \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}, \quad J_h^\alpha(y_h) := E_h^\alpha(y_h) - \frac{1}{h^\alpha} \int_\Omega f_h \cdot y_h \, dx,$$

where $f_h \colon S \to \mathbb{R}^3$ is a sequence of dead loads that satisfies

$$\frac{1}{h^{\gamma+1}} f_h \to f \quad \text{in } L^{q'}(S; \mathbb{R}^3). \tag{3.10}$$

Here q' is the conjugate exponent of q. Note that $q' \geq 2$, so that the strong convergence of the forces holds also in $L^2(S; \mathbb{R}^3)$. We assume the forces to be mean-free, i.e.,

$$\int_S f_h \, dx' = 0, \tag{3.11}$$

otherwise the infimum of J_h^α is $-\infty$. We prove the following result.

Theorem 3.1.6. *Suppose that S satisfies (2.17) and $q > 1$. Let $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations that are quasi-minimizers for J_h^2, i.e.,*

$$\limsup_{h\to 0}\,(J_h^2(y_h) - \inf J_h^2) = 0.$$

Then, $E_h^2(y_h) \leq C$ for every $h > 0$ and there exist $j \in \{1,\dots,l\}$ and $y \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$ such that, up to subsequences, $y_h \to y$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and (j,y) minimizes

$$J_j^{\mathrm{K}}(y) := E_j^{\mathrm{K}}(y) - \int_S f\cdot y\,dx,$$

over the set

$$\{(j,y) \in \{1,\dots,l\}\times W^{2,2}(S;\mathbb{R}^3)\colon \nabla' y^T \nabla' y = (U_j^2)'\}.$$

For the case $\alpha > 2$, optimal rotations play an important role (see Section 2.1 for the definition of optimal rotations and the related notation). We assume that the forces f_h are such that

(F^1) $\Lambda_h = \Lambda$ for $h \ll 1$,

(F^2) $\dim \mathcal{R}_h^j \to \dim \mathcal{R}^j$ for any $j \in \Lambda$.

Note that in general one only has $\Lambda_h \subseteq \Lambda$ and $\limsup_{h\to 0} \dim R_h^j \leq \dim R_h^j$, as shown in the following example. The failure of (F^1)–(F^2) may happen, for instance, when the direction along which the force acts is slightly perturbed.

Example 3.1.7. Let $\alpha > 2$ and set $S := (-1/2, 1/2)^2$. Suppose that $l := 2$ and let

$$U_1 := \mathrm{Id}, \qquad U_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We consider the following sequence of forces

$$f_h(x') := h^{\gamma+1}[x_1 e_3 + h x_2 e_2].$$

Note that the sequence f_h is mean-free by symmetry. Then, with some simple computation one has that

$$F_h(RU_1) = \frac{h^{\gamma+1}}{12}(R_{31} + hR_{22}),$$
$$F_h(RU_2) = \frac{h^{\gamma+1}}{12}(R_{31} + 2hR_{22}).$$

It follows that $\mathcal{R}_h^1 = \mathcal{R}_h^2$ are singletons given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and that $\Lambda_h = \{2\}$. The limit force f is given by $f(x') = x_1 e_3$. Hence,

$$F(RU_1) = F(RU_2) = \frac{1}{12} R_{31},$$

Thus,

$$\mathcal{R}^1 = \mathcal{R}^2 = \{R \in \mathrm{SO}(3) : R_{31} = 1\},$$

and $\dim \mathcal{R}^1 = \dim \mathcal{R}^2 = 1$. Moreover, $\Lambda = \{1, 2\}$.

Under the assumptions (F^1)–(F^2), the following result holds.

Theorem 3.1.8. *Let $\alpha > 2$ and $q > 1$. Assume (F^1)–(F^2). Suppose that $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ is a sequence of deformations that are quasi-minimizers for J_h^α, i.e.,*

$$\limsup_{h\to 0} \left(J_h^\alpha(y_h) - \inf J_h^\alpha\right) = 0.$$

Then, $E_h^\alpha(y_h) \le C$ and there exist $(\bar{R}_h) \subset \mathrm{SO}(3)$, $(c_h) \subset \mathbb{R}$, $u \in W^{1,2}(S;\mathbb{R}^2)$, $v \in W^{2,2}(S)$, and $j \in \{1, \dots, l\}$ such that $j \in \Lambda$ and, up to subsequences,

(i) *$\bar{R}_h \to \bar{R}$ with $\bar{R}U_j \in \mathcal{M}$,*

(ii) *$u_h \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$,*

(iii) *$v_h \to v$ in $W^{1,2}(\Omega)$,*

where u_h, v_h are defined as in (3.3)–(3.4) for $\tilde{y}_h := \bar{R}_h^T y_h + c_h$. In addition, there is $W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ such that $|W| = 1$, $\bar{R}W \in N\mathcal{R}^j_{\bar{R}}$, and

$$h^{\frac{1}{2}(1-\gamma)}(\bar{R}_h - P_h^j(\bar{R}_h)) \to \beta \bar{R} W,$$

for some $\beta \ge 0$. Lastly,

(a) *if $2 < \alpha < 4$ and S is a simply connected set that satisfies (2.17), then $(j, v, \bar{R}, \beta W)$ minimizes the functional*

$$J_j^{\mathrm{CVK}}(v, R, W) := E_j^{\mathrm{CVK}}(v) - \int_S f \cdot RU_j^{-1} e_3 v \, dx - F(RW^2 U_j)$$

over all the admissible quadruplets (j, v, R, W), that is, $j \in \Lambda$ and $(v, R, W) \in \mathcal{A}^{\mathrm{lin}}_{\mathrm{iso},j} \times \mathcal{R}^j \times \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ such that $W \in N\mathcal{R}^j_R$,

(b) *if $\alpha = 4$, then $(j, u, v, \bar{R}, \beta W)$ minimizes the functional*

$$J_j^{\mathrm{VK}}(u, v, R, W) := E_j^{\mathrm{VK}}(u, v) - \int_S f \cdot RU_j^{-1} e_3 v \, dx - F(RW^2 U_j)$$

over all the admissible quintuplet (j, u, v, R, W), that is, $j \in \Lambda$ and $(u, v, R, W) \in W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times \mathcal{R}^j \times \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ such that $W \in N\mathcal{R}^j_R$,

(c) *if $\alpha > 4$, then $(j, u, v, \bar{R}, \beta W)$ minimizes the functional*

$$J_j^{\mathrm{LVK}}(u, v, R, W) := E_j^{\mathrm{LVK}}(u, v) - \int_S f \cdot RU_j^{-1} e_3 v \, dx - F(RW^2 U_j)$$

over all the admissible quintuplets (j, u, v, R, W), that is, $j \in \Lambda$ and $(u, v, R, W) \in W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) \times \mathcal{R}^j \times \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ such that $W \in N\mathcal{R}^j_R$.

Remark 3.1.9. Given that the scaling of the forces is of order $h^{\gamma+1}$, we expect the action of the load on the in-plane displacement to be negligible. This is indeed the case and the limiting forcing term acts only on the out-of-plane displacement. The additional term $F(RW^2U_j)$ can be interpreted (see also [MM21]) as an elastic cost of fluctuations of the reference configuration from the optimal rotations.

From a minimization point of view, since the term $F(RW^2U_j)$ is always non-positive (see Section 2.2.3), it is clear that the optimal choice is $W = 0$. In particular, in Theorem 3.1.8 we actually have $\beta = 0$. Similarly, since for every $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and for every $j = 1, \dots, l$ we have

$$E_j^{\mathrm{LVK}}(Ax', v) < E_j^{\mathrm{LVK}}(u, v) \qquad \forall\, u \in W^{1,2}(S; \mathbb{R}^2) \text{ s.t. } \operatorname{sym}(\nabla u) \neq 0,$$

in Theorem 3.1.8–(c) we infer $u = Ax'$ for some $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$.

Before moving to the proofs of our results, we provide an example of rank-one connected double-well structure for which different applied forces result in different preferred reference configurations.

Example 3.1.10. Let $S := (-\frac{1}{2}, \frac{1}{2})^2$ and consider

$$U_1 := \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U_2 := \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

For $a, b, c \geq 0$, consider the sequences of loads

$$f_h(x') := h^{\gamma+1}(ax_1e_1 + bx_2e_2 + cx_1e_3).$$

The limit force $f(x') = ax_1e_1 + bx_2e_2 + cx_1e_3$ is pulling the mid-plane S along fibres parallel to e_1 and e_2, while twisting it in the out-of-plane direction. With some simple computation we get

$$\begin{aligned} F(RU_1) &= \frac{1}{12}(4aR_{11} + bR_{22} + cR_{31}), \\ F(RU_2) &= \frac{1}{12}[a(2R_{11} + R_{13}) + bR_{22} + c(2R_{31} + R_{33})]. \end{aligned}$$

Note that, if $a = 0$ and $b, c > 0$ (i.e., f is pulling the mid-plane S in the e_2 direction only), then one has that

$$F(RU_1) \leq \frac{1}{12}(b + c) < F(\bar{R}U_2), \qquad \forall\, R \in \mathrm{SO}(3),$$

where

$$\bar{R} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

In particular, $\Lambda = \{2\}$ and the only admissible well at the limit is $\mathrm{SO}(3)U_2$. However, if $a > 0$ and $b = c = 0$, that is f is tensing S along e_1 without twisting the mid-plane, we have

$$F(RU_2) = \frac{a}{12}(2R_{11} + R_{13}) \leq \frac{a}{4} < \frac{a}{3} = F(U_1), \qquad \forall\, R \in \mathrm{SO}(3).$$

Thus, $\Lambda = \{1\}$ and the only admissible reference configuration is $\mathrm{SO}(3)U_1$.

3.2 Compactness estimates

In order to study the Γ-convergence of the functionals E_h^α we first need to establish compactness for sequences of deformations that have bounded rescaled energy. It is clear that the elastic part of the rescaled energy forces the rescaled deformation gradient to approach in the limit the union of the wells K. However, we would like to prove that the rescaled gradients are actually getting closer to a single well K^i. This is precisely ensured by the penalty term. In the following result, inspired by [Ali+18], we give a precise meaning to this statement.

Proposition 3.2.1. *Let $\alpha \geq 2$. Let $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence such that*

$$\lim_{h\to 0} h^\alpha E_h^\alpha(y_h) = 0. \tag{3.12}$$

Then for $h \ll 1$ there are an index $i_h \in \{1,\dots,l\}$ and two constants $\delta, C(\delta) > 0$ such that

$$\int_{\Omega^{h,i_h}} \operatorname{dist}^2(\nabla_h y_h, K^{i_h})\,dx \leq C(\delta) h^\alpha E_h^\alpha(y_h), \tag{3.13}$$

$$\int_{\Omega\setminus\Omega^{h,i_h}} \operatorname{dist}^2(\nabla_h y_h, K^{i_h})\,dx \leq C(\delta) h^\alpha [(E_h^\alpha(y_h))^\theta + E_h^\alpha(y_h)], \tag{3.14}$$

where $\Omega^{h,i_h} := \{x \in \Omega : \operatorname{dist}(\nabla_h y_h(x), K^{i_h}) \leq \delta\}$ and

$$\theta := \begin{cases} 3/2 & \text{if } q = 2, \\ 5/3 & \text{otherwise.} \end{cases}$$

For the convenience of the reader, we give a self-contained proof following the same arguments used in [Ali+18, Theorem 2.3]. We start with two preliminary Lemmas, the proofs of which can be found in [Ali+18, Lemmas 2.5 and 2.6].

Define, for $A \in \mathbb{R}^{3\times 3}$,

$$\widetilde{W}(A) := f_q(\operatorname{dist}(A,K)),$$

where we recall that $f_q(t) := t^2 \wedge t^q$. Let

$$d_{\widetilde{W}}(A,B) := \inf\left\{ \int_0^1 (\widetilde{W}(\xi(s)))^{\frac{m_1}{m_2 p}} |\xi'(s)|\,ds \colon \xi \in C^1([0,1];\mathbb{R}^{d\times d}) \right.$$
$$\left. \text{s.t. } \xi(0) = A, \xi(1) = B \right\},$$

where $m_1, m_2 > 1$ are such that $1/m_1 + 1/m_2 = 1$.

Lemma 3.2.2. *Let $\delta > 0$ and $i \in \{0,\dots,l\}$. There exists $C > 0$, depending on δ, such that*

$$\operatorname{dist}(F,K^i) \leq C d_{\widetilde{W}}(F,K^i) \qquad \forall\, F \in \mathbb{R}^{3\times 3} \text{ s.t. } d_{\widetilde{W}}(F,K^i) \geq \delta.$$

Lemma 3.2.3. *Let $\delta > 0$ and $i \in \{0,\dots,l\}$. There exists a constant $C > 0$, depending on δ, such that*

$$\operatorname{dist}(F,K^i) \leq C \qquad \forall\, F \in \mathbb{R}^{3\times 3} \text{ s.t. } d_{\widetilde{W}}(F,K^i) < \delta.$$

Proof of Proposition 3.2.1. Fix $\delta < 1 \wedge \min_{i\neq j} \{ \operatorname{dist}(K^i, K^j)/2 \}$ and define

$$\Omega^h := \{x \in \Omega\colon \operatorname{dist}(\nabla_h y_h(x), K) \leq \delta\} = \bigcup_{i=1}^{l} \Omega^{h,i},$$

where

$$\Omega^{h,i} := \{x \in \Omega\colon \operatorname{dist}(\nabla_h y_h(x), K^i) \leq \delta\}.$$

Note that, by (3.1), condition (3.13) holds with any $j \in \{1, \dots, l\}$ in place of i_h. By (3.12) and (3.1) we have that

$$\begin{aligned} |\Omega \backslash \Omega^h| &= \frac{1}{f_q(\delta)} \int_{\Omega\backslash\Omega^h} f_q(\delta)\, dx \leq C(\delta) \int_{\Omega\backslash\Omega^h} f_q(\operatorname{dist}(\nabla_h y_h, K))\, dx \\ &\leq C(\delta) h^\alpha E_h(y_h) \to 0. \end{aligned} \tag{3.15}$$

Hence, there exists $i_h \in \{1, \dots, l\}$ such that $|\Omega^{h,i_h}| \geq C(\delta)$. We define

$$g_h(x) := (d_{\widetilde{W}}(\nabla_h y_h(x), K^{i_h}) - \tilde{\delta}) \vee 0,$$

where $\tilde{\delta} := \delta^{2\frac{m_1}{m_2 p}+1}$. For $x \in \Omega^{h,i_h}$ let $U_x \in K^{i_h}$ be such that

$$|\nabla_h y_h(x) - U_x| \leq \delta.$$

Let

$$\xi_x(t) := (1-t)\nabla_h y_h(x) + tU_x.$$

Recall that $A \mapsto \operatorname{dist}(A, K)$ is 1-Lipschitz and f_q is monotonically increasing. Thus

$$\widetilde{W}(\xi_x(t)) = f_q(\operatorname{dist}(\xi_x(t), K)) \leq f_q(|(1-t)\nabla_h y_h(x) - (1-t)U_x|) \leq f_q(\delta) = \delta^2.$$

We deduce that

$$d_{\widetilde{W}}(\nabla_h y_h(x), K^{i_h}) \leq \int_0^1 (\widetilde{W}(\xi_x(t)))^{\frac{m_1}{m_2 p}} |\nabla_h y_h(x) - U_x|\, dt \leq \delta^{2\frac{m_1}{m_2 p}+1} = \tilde{\delta}.$$

In particular, $g_h \equiv 0$ on Ω^{h,i_h}. Set $\beta := p/m_1$ and choose m_1 in such a way that $\beta < 3$. We prove that $g_h \in W^{1,\beta}(\Omega)$ and

$$\int_\Omega |\nabla g_h|^\beta\, dx \leq \int_\Omega (\widetilde{W}(\nabla_h y_h))^{\frac{1}{m_2}} |\nabla(\nabla_h y_h)|^\beta\, dx. \tag{3.16}$$

In order to do so, we proceed by approximation. Firstly, note that $d_{\widetilde{W}}(\cdot, K^{i_h})$ is locally Lipschitz. Indeed, since the triangular inequality holds for $d_{\widetilde{W}}$, we have

$$|d_{\widetilde{W}}(A, K^{i_h}) - d_{\widetilde{W}}(B, K^{i_h})| \leq d_{\widetilde{W}}(A, B) \leq \int_0^1 \widetilde{W}(tA + (1-t)B)^{\frac{m_1}{m_2 p}} |A - B|\, dt.$$

Thus, it is sufficient to estimate

$$\int_0^1 \widetilde{W}(tA + (1-t)B)^{\frac{m_1}{m_2 p}}\, dt = \int_0^1 f_q(\operatorname{dist}(tA + (1-t)B, K))^{\frac{m_1}{m_2 p}}\, dt,$$

when A, B belongs to a ball of radius R. Let $U_0 \in K$ be such that $\operatorname{dist}(0,K) = |U_0|$. Then, since $\operatorname{dist}(\cdot, K)$ is 1-Lipschitz and f_q is monotonically increasing

$$\begin{aligned}\int_0^1 f_q(\operatorname{dist}(tA+(1-t)B,K))^{\frac{m_1}{m_2 p}}\,dt &\le \int_0^1 f_q(|t(A-B)+B-U_0|)^{\frac{m_1}{m_2 p}}\,dt \\ &\le f_q(|A|+2|B|+\operatorname{dist}(0,K))^{\frac{m_1}{m_2 p}} \le C(R).\end{aligned}$$

Define

$$g_h^n := (d_{\widetilde{W}}(\zeta_n, K^{i_h}) - \tilde{\delta}) \vee 0,$$

with

$$\zeta_n := -n \vee (n \wedge \nabla_h y_h(x)),$$

where both $\vee$ and $\wedge$ are intended entrywise. Since $d_{\widetilde{W}}$ is locally lipschitz, and ζ_n is bounded, we deduce that $g_h^n \in L^\infty(\Omega)$. By classical results on composition of Lipschitz and Sobolev functions (see, for example, [Zie89, Theorem 2.1.11]), we deduce that $g_h^n \in W^{1,\infty}(\Omega)$ and the chain rule applies. In particular,

$$|\nabla g_h^n| \le |\nabla d_{\widetilde{W}}(\zeta_n, K^{i_h})||\nabla \zeta_n|.$$

For any $B \in \mathbb{R}^{3\times 3}$ with unitary norm, we have

$$|d_{\widetilde{W}}(A, K^{i_h}) - d_{\widetilde{W}}(A+\varepsilon B, K^{i_h})| \le d_{\widetilde{W}}(A, A+\varepsilon B) \le \int_0^1 \widetilde{W}(A+t\varepsilon B)^{\frac{m_1}{m_2 p}}|\varepsilon B|\,dt.$$

Dividing by ε and passing to the limit as $\varepsilon \to 0$, we deduce

$$|\nabla d_{\widetilde{W}}(A, K^{i_h})| \le \widetilde{W}(A)^{\frac{m_1}{m_2 p}}.$$

Thus, we can estimate

$$|\nabla d_{\widetilde{W}}(\zeta_n, K^{i_h})| \le C\widetilde{W}(\zeta_n)^{\frac{m_1}{m_2 p}}.$$

We show now that, at least for $n \gg 1$,

$$\widetilde{W}(\zeta_n) \le C\widetilde{W}(\nabla_h y_h). \tag{3.17}$$

Indeed, (3.17) is obvious if $\zeta_n = \nabla_h y_h$. If this is not the case, then $|\nabla_h y_h| \ge |\zeta_n|$, and for $n \gg 1$,

$$\begin{aligned}\operatorname{dist}(\zeta_n, K) &\le \operatorname{dist}(0,K) + |\zeta_n| \le \operatorname{dist}(0,K) + |\nabla_h y_h| \\ &\le \operatorname{dist}(0,K) + \max\{|U| : U \in K\} + \operatorname{dist}(\nabla_h y_h, K) \le 2\operatorname{dist}(\nabla_h y_h, K).\end{aligned}$$

Since f_q is monotonically increasing, (3.17) follows. Thus,

$$|\nabla g_h^n| \le C\widetilde{W}(\zeta_n)^{\frac{m_1}{m_2 p}}|\nabla\nabla_h y_h| \le C\widetilde{W}(\nabla_h y_h)^{\frac{m_1}{m_2 p}}|\nabla_h^2 y_h|,$$

so that by Young's inequality and (3.1)

$$|\nabla g_h^n|^\beta \le C\widetilde{W}(\nabla_h y_h)^{\frac{1}{m}}|\nabla_h^2 y_h|^{\frac{p}{m_1}} \le \widetilde{W}(\nabla_h y_h) + |\nabla_h^2 y_h|^p \le W(\nabla_h y_h) + |\nabla_h^2 y_h|^p.$$

Hence, ∇g_h^n is uniformly bounded in $L^\beta(\Omega,\mathbb{R}^3)$. Note that, for $n \gg 1$, we have $g_h^n \equiv 0$ on Ω^{h,i_h}. By Theorem 2.2.1, we get that g_h^n is uniformly bounded in $W^{1,\beta}(\Omega)$. Thus, up to subsequences, $g_h^n \rightharpoonup \tilde{g}_h$ in $W^{1,\beta}(\Omega)$. Since $g_h^n \to g_h$ poinwise, we have $\tilde{g}_h = g_h$. Estimate (3.16) follows by lower semicontinuity.

Applying the Sobolev's Embedding Theorem, Theorem 2.2.1, and the Young's inequality, we deduce that

$$\begin{aligned}\int_\Omega |g_h|^{\beta^*}\,dx &\le C(\delta)\left(\int_\Omega |\nabla g_h|^\beta\,dx\right)^{\frac{\beta^*}{\beta}}\\ &\le C(\delta)\left(\int_\Omega (\widetilde{\mathcal{W}}(\nabla_h y_h))^{\frac{1}{m_2}}|\nabla(\nabla_h y_h)|^\beta\,dx\right)^{\frac{\beta^*}{\beta}}\\ &\le C(\delta)\frac{h^{\alpha\frac{\beta^*}{\beta}}}{\eta(h)^{\beta^*}}E_h^\alpha(y_h)^{\frac{\beta^*}{\beta}},\end{aligned}\tag{3.18}$$

where $\beta^* := 3\beta/(3-\beta)$ is the critical Sobolev exponent in dimension 3. Note that here we used the crucial information that $|\Omega^{h,i_h}| \ge C(\delta)$ to deduce that the constant in the Poincaré's inequality can be chosen independently of h. Let

$$\widetilde{\Omega}^{h,i} := \{x \in \Omega\colon d_{\widetilde{\mathcal{W}}}(\nabla_h y_h, K^i) \le 2\tilde{\delta}\}.$$

By Lemma 3.2.3, we can refine the choice of δ in such a way that

$$\widetilde{\Omega}^{h,i_h}\backslash\Omega^{h,i_h} \subset \Omega\backslash\Omega^h \subset \Omega\backslash\Omega^{h,i_h}.$$

Moreover, by Lemmas 3.2.2 and 3.2.3 we have that

$$\begin{aligned}&\operatorname{dist}(\nabla_h y_h, K^{i_h}) \le C(\delta) g_h \quad \text{on } \Omega\setminus\widetilde{\Omega}^{h,i_h},\\ &\operatorname{dist}(\nabla_h y_h, K^{i_h}) \le C(\delta) \quad \text{on } \widetilde{\Omega}^{h,i_h}\setminus\Omega^{h,i_h}.\end{aligned}$$

Thus, writing $\Omega\setminus\Omega^{h,i_h}$ as $(\Omega\setminus\widetilde{\Omega}^{h,i_h})\cup(\widetilde{\Omega}^{h,i_h}\setminus\Omega^{h,i_h})$ we deduce from (3.15) and (3.18) that

$$\begin{aligned}\int_{\Omega\setminus\Omega^{h,i_h}}\operatorname{dist}^{\beta^*}(\nabla_h y_h, K^{i_h})\,dx &\le C(\delta)\int_{\Omega\setminus\widetilde{\Omega}^{h,i_h}}|g_h|^{\beta^*}\,dx + C(\delta)|\widetilde{\Omega}^{h,i_h}\setminus\Omega^{h,i_h}|\\ &\le C(\delta)\left(\frac{h^{\alpha\frac{\beta^*}{\beta}}}{\eta(h)^{\beta^*}}E_h^\alpha(y_h)^{\frac{\beta^*}{\beta}} + h^\alpha E_h^\alpha(y_h)\right).\end{aligned}\tag{3.19}$$

If $q \neq 2$, we choose $l := 5p/6$. Note that $m_1 > 1$ by (P3). Then $\beta = 6/5$, $\beta^* = 2$, and $\beta^*/\beta = 5/3 = \theta$. By (P1) and (3.19) we get (3.14).

If $q = 2$, we choose $m_1 := p$, so that $\beta = 1$, $\beta^* = 3/2$ and $\beta^*/\beta = \theta = 3/2$. Fix a constant $M > 0$ such that $K \subset B_M(0)$ and define

$$B_h^M := \{x \in \Omega\colon |\nabla_h y_h(x)| \le M\}.$$

Writing $\Omega\backslash\Omega^{h,i_h}$ as $((\Omega\backslash\Omega^{h,i_h})\cap B_h^M)\cup((\Omega\backslash\Omega^{h,i_h})\backslash B_h^M)$, by (3.1) we deduce that

$$\begin{aligned}\int_{\Omega\backslash\Omega^{h,i_h}}\operatorname{dist}^2(\nabla_h y_h, K^{i_h})\,dx &\le C(\delta)\int_{(\Omega\backslash\Omega^{h,i_h})\cap B_h^M}\operatorname{dist}^{\frac{3}{2}}(\nabla_h y_h, K^{i_h})\,dx\\ &\quad + C(\delta)\int_{(\Omega\backslash\Omega^{h,i_h})\backslash B_h^M}\mathcal{W}(\nabla_h y_h)\,dx.\end{aligned}\tag{3.20}$$

Then, (3.14) follows by (P1) and (3.19)–(3.20). □

Remark 3.2.4. At a first glance, it might seem that the hypothesis (3.12) of Proposition 3.2.1 does not depend on α, while the thesis (3.14) does. However, (P1) prescribes a dependence on α of the penalty term coefficient $\eta(h)$. This is particularly clear examining the proof above.

Corollary 3.2.5. *Let $\alpha \geq 2$. Let $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence such that*

$$\lim_{h\to 0} h^\alpha E_h^\alpha(y_h) = 0.$$

Then for $h \ll 1$ there are an index $i_h \in \{1,\dots,l\}$ and a constant $C > 0$ such that

$$\int_\Omega \operatorname{dist}^2(\nabla_h y_h, K^{i_h})\,dx \leq Ch^\alpha E_h^\alpha(y_h).$$

The following is a variant of the well-known rigidity estimate by Friesecke, James, and Müller (see [FJM02; FJM06]), where the well SO(3) is replaced by $K^j = \mathrm{SO}(3)U_j$ (see Section 2.1 for the definition of K^j).

Proposition 3.2.6. *Let $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ and let $j \in \{1,\dots,l\}$. Define*

$$D_{h,j} := \|\operatorname{dist}(\nabla_h y_h, K^j)\|_{L^2(\Omega)}.$$

There are two maps $R_h \in L^\infty(S;\mathrm{SO}(3))$ and $\tilde{R}_h \in W^{1,2}(S;\mathbb{R}^{3\times 3}) \cap L^\infty(S;\mathbb{R}^{3\times 3})$ such that

(R1) $\|\nabla_h y_h - R_h U_j\|_{L^2(\Omega)} \leq CD_{h,j}$,

(R2) $\|\nabla' \tilde{R}_h\|_{L^2(S)} \leq Ch^{-1}D_{h,j}$,

(R3) $\|\tilde{R}_h - R_h\|_{L^2(S)} \leq CD_{h,j}$,

(R4) $\|\tilde{R}_h - R_h\|_{L^\infty(S)} \leq Ch^{-1}D_{h,j}$.

Moreover, there exists a constant rotation $Q_h \in \mathrm{SO}(3)$ such that

$$\|R_h - Q_h\|_{L^2(S)} \leq Ch^{-1}D_{h,j}.$$

Finally, if $h^{-1}D_{h,j} \to 0$, then for $h \ll 1$ we can choose $\tilde{R}_h = R_h$.

To prove this result it is enough to follow the same approach of [FJM06]. Indeed, the rigidity estimate [FJM06, Theorem 5] holds also for a well of the form $\mathrm{SO}(3)U_j$ by a change of variable. Then, all the estimates in [FJM06, Theorem 6] can be carried out in the same fashion.

Remark 3.2.7. If $r > 1$, all the results of Proposition 3.2.6 hold with the L^2 norm replaced by the L^r norm and the factor h^{-1} replaced by $h^{-2/r}$

3.2.1 Compactness in the Kirchhoff's regime

In the case $\alpha = 2$ the Γ-limit is written in terms of the deformation gradient. In this section we show compactness for sequences of rescaled gradients and give a characterization of their limit. Firstly, we need an explicit expression of the vector ν that appears in (3.2).

Lemma 3.2.8. *Let U be a symmetric and positive definite matrix. Let $y \in W^{2,2}(S;\mathbb{R}^3)$ be such that $\nabla' y^T \nabla' y = (U^2)'$. Then there exists a unique function $\nu \in W^{1,2}(S;\mathbb{R}^3)$ such that*

$$\left(\nabla' y \quad \nu\right) U^{-1} \in SO(3) \quad \textit{a.e. in } S.$$

In particular, ν is given by

$$\nu = \frac{1}{|U^{-1}e_3|^2}\left[\det(U^{-1})(\partial_1 y \wedge \partial_2 y) - \sum_{k=1}^{2}(U^{-1}e_k \cdot U^{-1}e_3)\partial_k y\right].$$

Proof. For the existence, it is enough to prove that $\left(\nabla' y \quad \nu\right)^T \left(\nabla' y \quad \nu\right) = U^2$ and $\det\left(\nabla' y \quad \nu\right) > 0$. By the hypothesis on y we need to prove that

(i) $\partial_1 y \cdot \nu = (U^2)_{13}$,

(ii) $\partial_2 y \cdot \nu = (U^2)_{23}$,

(iii) $\nu \cdot \nu = (U^2)_{33}$.

For $j = 1,2$ we have

$$\partial_j y \cdot \nu = -\frac{1}{|U^{-1}e_3|^2}\sum_{k=1}^{3}(U^{-1}e_k \cdot U^{-1}e_3)(U^2)_{jk} + (U^2)_{j3} = (U^2)_{j3}.$$

To complete the proof we observe that

$$|Ue_1 \wedge Ue_2|^2 = |Ue_1|^2|Ue_2|^2 - (Ue_1 \cdot Ue_2)^2 = |\partial_1 y \wedge \partial_2 y|^2$$

and, since $Ue_1 \wedge Ue_2 = \mathrm{cof}(U)e_3$

$$|Ue_1 \wedge Ue_2|^2 = |\det(U)U^{-1}e_3|^2 = \det{}^2(U)|U^{-1}e_3|^2.$$

We are now ready to conclude:

$$\begin{aligned}
\nu \cdot \nu = {} & \frac{1}{|U^{-1}e_3|^2} + \frac{(U^{-1}e_1 \cdot U^{-1}e_3)}{|U^{-1}e_3|^4}\sum_{k=1}^{3}(U^{-1}e_k \cdot U^{-1}e_3)(U^2)_{1k} \\
& + \frac{(U^{-1}e_2 \cdot U^{-1}e_3)}{|U^{-1}e_3|^4}\sum_{k=1}^{3}(U^{-1}e_k \cdot U^{-1}e_3)(U^2)_{2k} \\
& - \frac{1}{|U^{-1}e_3|^2}\sum_{k=1}^{3}(U^{-1}e_k \cdot U^{-1}e_3)(U^2)_{3k} + (U^2)_{33} \\
= {} & \frac{1}{|U^{-1}e_3|^2} - \frac{1}{|U^{-1}e_3|^2} + (U^2)_{33} = (U^2)_{33}.
\end{aligned}$$

To show that $\det \begin{pmatrix} \nabla' y & \nu \end{pmatrix} > 0$, it is sufficient to note that

$$\begin{pmatrix} \nabla' y & \nu \end{pmatrix} = \begin{pmatrix} \partial_1 y & \partial_2 y & \partial_1 y \wedge \partial_2 y \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{1}{|U^{-1}e_3|^2}(U^{-1}e_1 \cdot U^{-1}e_3) \\ 0 & 1 & -\frac{1}{|U^{-1}e_3|^2}(U^{-1}e_2 \cdot U^{-1}e_3) \\ 0 & 0 & \frac{\det(U^{-1})}{|U^{-1}e_3|^2} \end{pmatrix},$$

and that the determinant of both matrices in the right-hand side is positive.

To prove uniqueness we observe that, for any choice of two different rotations $R_1, R_2 \in \mathrm{SO}(3)$, we have $\operatorname{rank}(R_1 - R_2) \neq 1$. Indeed, given a vector $v \in \mathbb{R}^3$ we have

$$(R_1 - R_2)v = 0 \iff R_1 v = R_2 v \iff R_2^T R_1 v = v.$$

Since $R_2^T R_1 \in \mathrm{SO}(3)$ and $R_2^T R_1 \neq \mathrm{Id}$, we deduce that $v \in \ker(R_1 - R_2)$ if and only if v is parallel to the rotation axis, that is, v belongs to a 1-dimensional subspace. In particular, $\operatorname{rank}(R_1 - R_2) = 2$. Suppose now that there is another vector $\tilde{\nu}$ such that

$$\begin{pmatrix} \nabla' y & \tilde{\nu} \end{pmatrix} U^{-1} \in \mathrm{SO}(3) \text{ a.e..}$$

Then,

$$\begin{pmatrix} 0 & \nu - \tilde{\nu} \end{pmatrix} U^{-1} = (\nu - \tilde{\nu}) \otimes U^{-1} e_3$$

coincides almost everywhere with the difference of two rotations and has rank 1 whenever $\nu \neq \tilde{\nu}$. Thus, $\nu = \tilde{\nu}$ almost everywhere. □

We move now to the proof of the first part of Theorem 3.1.1.

Proof of Theorem 3.1.1–(i). By Corollary 3.2.5 there is a sequence of indices $i_h \in \{1, \dots, l\}$ such that

$$\| \operatorname{dist}(\nabla_h y_h, K^{i_h}) \|_{L^2} \leq Ch.$$

Upon a further subsequence, since i_h takes values in a finite set, we can suppose i_h to be constant and equal to j. Construct the sequences R_h and $\tilde{R}_h$ as in Proposition 3.2.6. Clearly, $\tilde{R}_h$ is bounded in $W^{1,2}(S; \mathbb{R}^{3\times 3})$ thus it converges weakly, at least along a subsequence, to a map $R \in W^{1,2}(S; \mathbb{R}^{3\times 3})$. Hence, we have $R_h \to R$ in $L^2(S; \mathrm{SO}(3))$, so R takes values in the set of rotations. Consequently, $\nabla_h y_h U_j^{-1} \to R$ in $L^2(S; \mathbb{R}^{3\times 3})$. By an application of the Poincaré–Wirtinger inequality we have

$$\|y_h - c_h\|_{W^{1,2}} \leq C \|\nabla_h y_h\|_{L^2} \leq C,$$

where

$$c_h := \frac{1}{|\Omega|} \int_\Omega y_h(x)\, dx.$$

Thus, $y_h - c_h$ converges weakly (possibly along a subsequence) to some map $y \in W^{1,2}(\Omega; \mathbb{R}^3)$. Since $h^{-1}\partial_3 y_h$ is bounded in $L^2(\Omega; \mathbb{R}^3)$, we have $\partial_3 y = 0$ and $y \in W^{1,2}(S; \mathbb{R}^3)$. Hence, $\nabla' y = (RU_j)^{1,2}$, $y \in W^{2,2}(S; \mathbb{R}^3)$ and $\nabla'_h y_h \to \nabla' y$. Lastly, since ν is uniquely determined by the condition $\begin{pmatrix} \nabla' y & \nu \end{pmatrix} U_j^{-1} \in \mathrm{SO}(3)$ almost everywhere, the remaining part follows from Lemma 3.2.8. □

Corollary 3.2.9. *In the same setting of Theorem 3.1.1, there is a sequence* $(R_h) \subset L^\infty(S;\mathrm{SO}(3))$ *such that, up to a subsequence*

$$G_h := h^{-1}\left(R_h^T \nabla_h y_h - U_j\right) \rightharpoonup G \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{3.21}$$

Moreover, $G^{1,2}$ *is affine in* x_3*, that is*

$$G^{1,2}(x',x_3) = G_a(x') + x_3 G_b(x').$$

Lastly,

$$(U_j G_b)' = \nabla' y^T \nabla' \nu. \tag{3.22}$$

Proof. Arguing as in the proof of Theorem 3.1.1–(i) we have

$$\|\operatorname{dist}(\nabla_h y_h, K^j)\|_{L^2} \le Ch,$$

for some $j \in \{1,\dots,l\}$. Let $R_h \in L^\infty(S;\mathrm{SO}(3))$ be the map given by Proposition 3.2.6. Convergence (3.21) follows from (R1). Moreover, arguing again as in the proof of Theorem 3.1.1–(i) we deduce that, up to subsequences, $R_h \to R \in L^2(S;\mathrm{SO}(3))$ and $\nabla_h y_h U_j^{-1} \to (\nabla' y \quad \nu)\, U_j^{-1} = R$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$, where ν is given by Lemma 3.2.8. Define

$$H_h^s(x',x_3) := \frac{1}{s}(G_h(x',x_3+s) - G_h(x',x_3)),$$

for s such that $x_3 + s \in I$. For $\alpha = 1,2,3$ and $\beta = 1,2$ we have

$$\begin{aligned}(R_h(x')H_h^s(x',x_3))_{\alpha\beta} &= \frac{1}{s}h^{-1}\left(\frac{\partial y_{h,\alpha}}{\partial x_\beta}(x',x_3+s) - \frac{\partial y_{h,\alpha}}{\partial x_\beta}(x',x_3)\right)\\ &= \frac{1}{s}\frac{\partial}{\partial x_\beta}\int_0^s \frac{1}{h}\frac{\partial y_{h,\alpha}}{\partial x_3}(x',x_3+\sigma)\,d\sigma.\end{aligned}$$

The right-hand side converges strongly in $(W_0^{1,2}(\Omega))^*$ to $(\nabla'\nu)_{\alpha\beta}$ as $h \to 0$. Indeed, one has that $\|\partial_i g\|_{(W_0^{1,2})^*} \le \|g\|_{L^2}$ for every $g \in L^2(\Omega)$, where $\|\cdot\|_{(W_0^{1,2})^*}$ is the standard operatorial norm. The left-hand side converges weakly in $L^2(\Omega)$ to

$$\left((\nabla' y \quad \nu)\, U_j^{-1} H^s(x',x_3)\right)_{\alpha\beta},$$

where

$$H^s(x',x_3)_{\alpha\beta} := \frac{1}{s}(G(x',x_3+s)_{\alpha\beta} - G(x',x_3)_{\alpha\beta}).$$

Since $L^2(\Omega)$ is continuously embedded in $(W_0^{1,2}(\Omega))^*$ we obtain

$$H^s(x',x_3)_{\alpha\beta} = \left(U_j^{-1}\,(\nabla' y \quad \nu)^T\, \nabla'\nu\right)_{\alpha\beta} \qquad \forall\, \alpha = 1,2,3,\ \forall\, \beta = 1,2.$$

In particular, the first two columns of H^s are independent of x_3 and so the first two columns of G are affine in x_3. Finally, we have

$$(U_j G_b)' = ((\nabla' y \quad \nu)^T\, \nabla'\nu)_{1,2} = \nabla' y^T \nabla'\nu,$$

that proves (3.22). ☐

3.2.2 Compactness in the Von Kármán's regime

In the case $\alpha > 2$ we write the Γ-limit as a function of the in-plane and out-of-plane displacements. The next results give the correct scaling to extract their convergence. We start with a preliminary Lemma.

Lemma 3.2.10. *Let $(M_h) \subset \mathbb{R}^{3\times3}$ be a sequence such that $|M_h - \mathrm{Id}\,| \leq Ch^\beta$ for some $\beta > 0$. Then there is a sequence $(P_h) \subset \mathrm{SO}(3)$ such that $\mathrm{skew}(P_h M_h) = 0$ and $|P_h - \mathrm{Id}\,| \leq Ch^\beta$.*

Proof. For $h \ll 1$ the matrix M_h is invertible with positive determinant and so its polar decomposition $M_h = R_h A_h$ is uniquely determined providing a matrix $P_h \in \mathrm{SO}(3)$ (i.e., $P_h = R_h^T$) such that $P_h M_h$ is symmetric. We have that

$$|M_h - R_h| \leq |M_h - X| \qquad \forall\, X \in \mathrm{O}(3). \tag{3.23}$$

Indeed, since $M_h = R_h A_h$, one has

$$|M_h - R_h|^2 = \mathrm{Tr}((M_h - R_h)^T(M_h - R_h)) = |M_h|^2 - 2\,\mathrm{Tr}(A_h) + 3,$$

and similarly

$$|M_h - X|^2 = |M_h|^2 - 2\,\mathrm{Tr}(M_h^T X) + 3.$$

Since A_h is symmetric and positive definite we can write $A_h = O_h \Sigma_h O_h^T$ for some $O_h \in \mathrm{O}(3)$ and $\Sigma_h := \mathrm{diag}(\sigma_h)$, with $\sigma_{h,i} > 0$. Thus, since the trace is invariant under circular shifts

$$\begin{aligned}
|M_h - R_h|^2 - |M_h - X|^2 &= 2\,\mathrm{Tr}(M_h^T X - A_h) = 2\,\mathrm{Tr}(O_h \Sigma_h O_h^T R_h^T X - O_h \Sigma_h O_h^T) \\
&= 2\,\mathrm{Tr}(\Sigma_h O_h^T R_h^T X O_h - \Sigma_h) = 2\,\mathrm{Tr}(\Sigma_h Y_h - \Sigma_h) \\
&= 2\sum_{i=1}^{3} \sigma_{h,i}((Y_h)_{ii} - 1) \leq 0,
\end{aligned}$$

where $Y_h := O_h^T R_h^T X O_h \in \mathrm{O}(3)$. Hence, by (3.23),

$$|P_h - \mathrm{Id}\,| = |R_h - \mathrm{Id}\,| \leq |R_h - M_h| + |M_h - \mathrm{Id}\,| \leq Ch^\beta.$$

$\square$

Proposition 3.2.11. *Let $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations such that*

$$\limsup_{h\to0} E_h^\alpha(y_h) \leq C,$$

where $\alpha > 2$. Then, for $h \ll 1$ there are an index $j \in \{1,\dots,l\}$, rotations $\bar{R}_h \in \mathrm{SO}(3)$, and vectors $c_h \in \mathbb{R}$ such that, setting $\tilde{y}_h$ as follows

$$\tilde{y}_h := \bar{R}_h^T y_h + c_h,$$

there exist $\tilde{R}_h \in W^{1,2}(S,\mathrm{SO}(3))$ that satisfies:

$$\|\nabla_h \tilde{y}_h - \tilde{R}_h U_j\|_{L^2(\Omega)} \leq Ch^\gamma, \tag{3.24}$$

$$\|\tilde{R}_h - \mathrm{Id}\,\|_{L^2(S)} \leq Ch^{\gamma-1}, \tag{3.25}$$

$$\|\nabla' \tilde{R}_h\|_{L^2(S)} \leq Ch^{\gamma-1}. \tag{3.26}$$

Moreover, there exists $A \in W^{1,2}(S, \mathbb{R}^{3\times 3}_{\mathrm{skew}})$ such that the following convergences hold true, possibly along a nonrelabelled subsequence:

$$A_h := h^{1-\gamma}(\tilde{R}_h - \mathrm{Id}) \rightharpoonup A \qquad \text{in } W^{1,2}(S;\mathbb{R}^{3\times 3}), \tag{3.27}$$

$$h^{1-\gamma}(\nabla_h \tilde{y}_h - U_j) \to AU_j \qquad \text{in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{3.28}$$

$$h^{2-2\gamma}\,\mathrm{sym}(\tilde{R}_h - \mathrm{Id}) \to \frac{A^2}{2} \qquad \text{in } L^2(S;\mathbb{R}^{3\times 3}). \tag{3.29}$$

Lastly, convergences (a)-(b) of Theorem 3.1.3–(i) hold and A has the following structure:

$$U_j A U_j = e_3 \otimes \nabla' v - \nabla' v \otimes e_3. \tag{3.30}$$

Proof. Up to a subsequence extraction, by Corollary 3.2.5 we have, for some index $j \in \{1, \dots, l\}$, that

$$D_{h,j} := \|\,\mathrm{dist}(\nabla_h y_h, K^j)\|_{L^2(\Omega)} \leq Ch^{\gamma},$$

where we recall that $\gamma := \alpha/2$. Let Q_h and R_h be, respectively, the constant rotation and the map whose existence is guaranteed by Proposition 3.2.6. Since $h^{-1}D_{h,j} \to 0$ we can suppose that $R_h \in W^{1,2}(S;\mathrm{SO}(3))$. Let $\tilde{y}_h := Q_h^T y_h$. By definition $\tilde{y}_h$ and $\tilde{R}_h := Q_h^T R_h$ satisfy (3.24), (3.25) and (3.26). Let

$$M_h := \frac{1}{|\Omega|}\int_\Omega \nabla_h \tilde{y}_h U_j^{-1}\,dx.$$

From (3.24) and (3.25) we can deduce

$$\begin{aligned}|M_h - \mathrm{Id}\,| &= \left| M_h - \frac{1}{|\Omega|}\int_\Omega \mathrm{Id}\,dx \right| \leq \frac{1}{|\Omega|}\int_\Omega |\nabla_h \tilde{y}_h U_j^{-1} - \mathrm{Id}\,|\,dx \\ &\leq Ch^{\gamma-1}.\end{aligned}$$

Therefore, by Lemma 3.2.10 there exists a rotation $P_h \in \mathrm{SO}(3)$ such that $|P_h - \mathrm{Id}\,| \leq Ch^{\gamma-1}$ and

$$\int_\Omega \mathrm{skew}\left(P_h \nabla_h \tilde{y}_h U_j^{-1}\right)\,dx = 0.$$

Thus, redefining $\tilde{y}_h := P_h Q_h^T y_h$ (so that $\bar{R}_h := Q_h P_h^T$) and $\tilde{R}_h = P_h Q_h^T R_h$ we can additionally suppose that

$$\int_\Omega \mathrm{skew}\left(\nabla_h \tilde{y}_h U_j^{-1}\right)\,dx = 0. \tag{3.31}$$

Moreover, we can choose the additive constant vector c_h so that

$$\int_\Omega \left(\tilde{y}_h - U_j\begin{pmatrix} x' \\ hx_3\end{pmatrix}\right)\,dx = 0. \tag{3.32}$$

From (3.25) we deduce that $\|A_h\|_{L^2} \leq C$. Moreover, since $\nabla A_h = h^{1-\gamma}\nabla \tilde{R}_h$, by (3.26) we have that A_h is bounded in $W^{1,2}(S;\mathbb{R}^{3\times 3})$. Hence, up to a subsequence, there is $A \in W^{1,2}(S;\mathbb{R}^{3\times 3})$ such that (3.27) holds true. By (3.24) and (3.27) we get (3.28). Using the identity

$$(Q - \mathrm{Id})^T(Q - \mathrm{Id}) = -2\,\mathrm{sym}(Q - \mathrm{Id}) \qquad \forall\, Q \in \mathrm{SO}(3),$$

we obtain

$$\operatorname{sym}(A_h) = \frac{1}{2} h^{1-\gamma} (\tilde{R}_h - \mathrm{Id})^T (\tilde{R}_h - \mathrm{Id}) \to 0 = \operatorname{sym}(A) \quad \text{in } L^2(S; \mathbb{R}^{3\times 3}),$$

that is, $A = -A^T$. In particular, we have

$$h^{2-2\gamma} \operatorname{sym}(\tilde{R}_h - \mathrm{Id}) = -\frac{1}{2} A_h^T A_h \to \frac{A^2}{2} \quad \text{in } L^2(S; \mathbb{R}^{3\times 3}),$$

that is precisely (3.29).

To simplify the notation, we will write U instead of U_j for the rest of the proof. Consider v_h as defined in (3.4) for the deformation $\tilde{y}_h$. By (3.32) and the Poincaré–Wirtinger inequality it follows that

$$\begin{aligned} \|v_h\|^2_{W^{1,2}(S)} &\le C \|\nabla v_h\|^2_{L^2(S;\mathbb{R}^2)} \le C h^{2-2\gamma} \int_\Omega |U(\nabla' \tilde{y}_h(x) - U^{1,2})|^2 \, dx \\ &\le C h^{2-2\gamma} \int_\Omega |\nabla'_h \tilde{y}_h - U^{1,2}|^2 \, dx \le C h^{2-2\gamma} \int_\Omega |\nabla_h \tilde{y}_h - U|^2 \, dx \le C. \end{aligned}$$

Hence, up to a subsequence, there is a map $v \in W^{1,2}(S)$ such that $v_h \rightharpoonup v$ in $W^{1,2}(S)$. Now, we want to show that $\nabla' v_h \to \nabla' v = e_3^T (UAU)^{1,2}$ from which we can deduce that $v \in W^{2,2}(S)$. Clearly,

$$\nabla' v_h = h^{1-\gamma} \int_\Omega e_3^T U \left(\nabla' \tilde{y}_h - U^{1,2} \right) \, dx.$$

By (3.28), it follows that

$$\nabla' v_h \to \nabla' v = e_3^T (UAU)^{1,2}. \tag{3.33}$$

Now we focus on the map u_h defined as in (3.3). We have

$$\begin{aligned} \int_\Omega \operatorname{skew}(\nabla' u_h) \, dx &= \min\left\{ h^{-\gamma}, h^{2-2\gamma} \right\} \int_\Omega \operatorname{skew}\left(U_{1,2} (\nabla' \tilde{y}_h - U^{1,2}) \right) \, dx \\ &= \min\left\{ h^{-\gamma}, h^{2-2\gamma} \right\} \int_\Omega U_{1,2} \operatorname{skew}\left(\nabla_h \tilde{y}_h U^{-1} - \mathrm{Id} \right) U^{1,2} \, dx \\ &= \min\left\{ h^{-\gamma}, h^{2-2\gamma} \right\} U_{1,2} \left[\int_\Omega \operatorname{skew}\left(\nabla_h \tilde{y}_h U^{-1} - \mathrm{Id} \right) \, dx \right] U^{1,2} \end{aligned}$$

and the last term is identically zero by (3.31). Therefore, we can apply Korn's inequality to deduce

$$\begin{aligned} \|u_h\|^2_{W^{1,2}} &\le C \|\operatorname{sym}(\nabla' u_h)\|^2_{L^2(S;\mathbb{R}^{2\times 2})} \\ &\le C \min\left\{ h^{-2\gamma}, h^{4-4\gamma} \right\} \int_\Omega \left| \operatorname{sym}(\nabla_h \tilde{y}_h U^{-1} - \mathrm{Id}) \right|^2 \, dx \\ &\le C \min\left\{ h^{-2\gamma}, h^{4-4\gamma} \right\} \left[\|\nabla_h \tilde{y}_h U^{-1} - \tilde{R}_h\|^2_{L^2} + \|\operatorname{sym}(\tilde{R}_h - \mathrm{Id})\|^2_{L^2} \right] \\ &\le C \min\left\{ h^{-2\gamma}, h^{4-4\gamma} \right\} \max\left\{ h^{2\gamma}, h^{4\gamma-4} \right\} \le C, \end{aligned}$$

where we have used (3.24) and (3.29). This proves the weak convergence of u_h, up to subsequences. Reasoning as before it is easy to note that

$$h^{1-\gamma} \max\left\{ h^{\gamma}, h^{2\gamma-2} \right\} \nabla' u_h \to U_{1,2} A U^{1,2} = (UAU)'.$$

By the assumption $\gamma = \alpha/2 > 1$ we have that $h^{1-\gamma}\max\{h^\gamma, h^{2\gamma-2}\} \to 0$, so $(UAU)' = 0$. Since A is skew-symmetric, so is UAU and this shows (3.30) by (3.33). □

Remark 3.2.12. In our setting u_h and v_h are the (suitably rescaled) components of the displacement ω_h with respect to the reference configuration $U_j\Omega$ in the basis $\{U_j^{-1}e_i : i = 1,2,3\}$, that is,

$$\omega_h = U_j^{-1}\begin{pmatrix}\max\{h^\gamma, h^{2\gamma-2}\}u_h \\ h^{\gamma-1}v_h\end{pmatrix}.$$

Note that a basis of the tangent space to the midplane $U_j(S\times\{0\})$ is given by $\{U_je_1, U_je_2\}$, while the normal direction is $U_j^{-1}e_3$. Thus, it may look more natural to define the in-plane and the out-of-plane displacement in terms of this basis, that is, to consider $\tilde{u}_h$ and $\tilde{v}_h$ such that

$$\omega_h = U_j\begin{pmatrix}\tilde{u}_h \\ 0\end{pmatrix} + \tilde{v}_h U_j^{-1}e_3.$$

It is easy to see that

$$\begin{aligned} u_h &= \min\{h^{-\gamma}, h^{2-2\gamma}\}(U_j^2)'\tilde{u}_h, \\ v_h &= h^{1-\gamma}\left(\tilde{v}_h + (U_j^2)^{1,2}\begin{pmatrix}\tilde{u}_h \\ 0\end{pmatrix}\cdot e_3\right), \end{aligned}$$

so that $h^{1-\gamma}\tilde{v}_h$ has the same limit as v_h, while $\min\{h^{-\gamma}, h^{2-2\gamma}\}\tilde{u}_h$ converges to some $\tilde{u}$, representing the same displacement as u expressed in a different basis. Note that the same argument would apply defining $\tilde{u}_h$ in terms of a basis of the form $\{v_1, v_2, U_j^{-1}e_3\}$, with $v_1, v_2 \in \operatorname{span}\{U_je_1, U_je_2\}$.

Corollary 3.2.13. *In the same notation and hypothesis of Proposition 3.2.11, there exists a map $G \in L^2(\Omega, \mathbb{R}^{3\times3})$ such that, up to a subsequence,*

$$G_h := h^{-\gamma}\left(\tilde{R}_h^T\nabla_h\tilde{y}_h - U_j\right) \rightharpoonup G \text{ in } L^2(\Omega;\mathbb{R}^{3\times3}). \tag{3.34}$$

Moreover, $G^{1,2}$ is affine in x_3, that is

$$G^{1,2}(x', x_3) = G_a(x') + x_3G_b(x').$$

Finally,

$$(U_jG_b)' = -(\nabla')^2 v$$

and

$$\operatorname{sym}(U_jG_a)' = \operatorname{sym}(\nabla' u) \qquad \text{if } \alpha > 4, \tag{3.35}$$

$$(\operatorname{sym}(U_jG_a))' = \operatorname{sym}(\nabla' u) + \frac{1}{2}|U_j^{-1}e_3|^2\nabla' v\otimes\nabla' v \qquad \text{if } \alpha = 4, \tag{3.36}$$

$$\nabla' u + \nabla' u^T + |U_j^{-1}e_3|^2\nabla' v\otimes\nabla' v = 0 \qquad \text{if } 2 < \alpha < 4. \tag{3.37}$$

Proof. We will write U in place of U_j to simplify the notation. Convergence (3.34) follows immediately from (3.24). To show that $G^{1,2}$ is affine, we study the difference quotient

$$H_h^s(x', x_3) := \frac{1}{s}(G_h(x', x_3 + s) - G_h(x', x_3)),$$

for s such that $x_3 + s \in I$. Repeating the same computation as in Corollary 3.2.9, we deduce that for $\alpha = 1, 2, 3$ and $\beta = 1, 2$ we have

$$(\tilde{R}_h(x')H_h^s(x', x_3))_{\alpha\beta} = \frac{1}{s}\frac{\partial}{\partial x_\beta}\int_0^s h^{1-\gamma}\left(\frac{1}{h}\frac{\partial \tilde{y}_{h,\alpha}}{\partial x_3}(x', x_3 + \sigma) - E_{\alpha 3}\right) d\sigma. \tag{3.38}$$

By (3.28), the integral on the right hand-side converges strongly as $h \to 0$ in $L^2(\Omega)$ to

$$\int_0^s (AU)_{\alpha 3}\, d\sigma = s(AU)_{\alpha 3}.$$

Hence, the right-hand side of (3.38) converges strongly in the dual of $W_0^{1,2}(\Omega)$ to $\dfrac{\partial}{\partial x_\beta}[(AU)_{\alpha 3}]$. By (3.34), the left-hand side of (3.38) converges weakly in $L^2(\Omega)$ to

$$H^s(x', x_3)_{\alpha\beta} := \frac{1}{s}(G(x', x_3 + s)_{\alpha\beta} - G(x', x_3)_{\alpha\beta}). \tag{3.39}$$

Since $L^2(\Omega)$ is continuously embedded in $(W_0^{1,2}(\Omega))^*$, we obtain

$$H^s(x', x_3)_{\alpha\beta} = \frac{\partial}{\partial x_\beta}[(AU)_{\alpha 3}] \qquad \forall\, \alpha = 1, 2, 3,\ \forall\, \beta = 1, 2. \tag{3.40}$$

In particular, the first two columns of H^s are independent of x_3 (recall that A depends only on x') and so the first two columns of G are affine in x_3.

For the final part of the statement note that G_h can be rewritten as follows

$$G_h = \frac{\nabla_h \tilde{y}_h - U_j}{h^\gamma} - \frac{\tilde{R}_h U_j - U_j}{h^\gamma} + (\tilde{R}_h - \mathrm{Id})^T \frac{\nabla_h \tilde{y}_h - \tilde{R}_h U_j}{h^\gamma}.$$

Hence,

$$\begin{aligned}(U_j G_h)' = (U_j)_{1,2} \frac{\nabla_h \tilde{y}_h U_j^{-1} - \mathrm{Id}}{h^\gamma}(U_j)^{1,2} - h^{\gamma-2}(U_j)_{1,2}\frac{\tilde{R}_h - \mathrm{Id}}{h^{2\gamma-2}}(U_j)^{1,2} \\ + \left[U_j(\tilde{R}_h - \mathrm{Id})^T \frac{\nabla_h \tilde{y}_h - \tilde{R}_h U_j}{h^\gamma}\right]'.\end{aligned} \tag{3.41}$$

If $\gamma > 2$, then $\min\{h^{-\gamma}, h^{2-2\gamma}\} = h^{-\gamma}$. Integrating with respect to x_3, taking the symmetric part and passing to the limit as $h \to 0$ in view of (3.24), (3.27) and (3.29), we get (3.35).

If $\gamma = 2$, passing to the limit in (3.41) we also get the term $-(UA^2U')/2$. By the characterization (3.30) of UAU we get with some computation

$$\begin{aligned}(UA^2U)' &= (UAU(U^{-1})^2 UAU)' = (UAU)_{1,2}(U^{-1})^2(UAU)^{1,2} \\ &= -\nabla' v^T (U^{-1}e_3)^T U^{-1} e_3 \nabla' v = -|U^{-1}e_3|^2 \nabla' v \otimes \nabla' v,\end{aligned}$$

proving (3.36). Lastly, if $1 < \gamma < 2$, we multiply both sides of (3.41) by $h^{2-\gamma}$ before passing to the limit. The left-hand side converges to 0, while the right-hand side converges again to

$$\operatorname{sym}(\nabla' u) + \frac{1}{2}|U_j^{-1}e_3|^2 \nabla' v \otimes \nabla' v,$$

proving (3.37). Finally, note that $G_b = H^1$ by (3.39). Thus, by (3.40) for $\alpha, \beta = 1, 2$

$$(UG_b)_{\alpha\beta} = \sum_{k=1}^{3} (U)_{\alpha k} \sum_{l=1}^{3} \frac{\partial A_{kl}}{\partial x_\beta}(U)_{l3} = \frac{\partial}{\partial x_\beta}(UAU)_{\alpha 3} = -\frac{\partial^2 v}{\partial x_\beta \partial x_\alpha}.$$

□

Proof of Theorem 3.1.3–(i). It immediately follows from Proposition 3.2.11 and Corollary 3.2.13. □

3.3 Proof of Γ-convergence

We are now ready to complete the proofs of Theorem 3.1.1 and Theorem 3.1.3. By the results of the previous section, we just need to prove the lim inf inequality and the existence of recovery sequences.

3.3.1 The lim inf inequality

Proof of Theorem 3.1.1–(ii) and Theorem 3.1.3–(ii). Define the matrix G_h as in Corollary 3.2.13 or Corollary 3.2.9, depending on the value of α. Let

$$\tilde{\Omega}_h := \{x \in \Omega : |G_h(x)| < h^{-1}\}$$

and let χ_h be its characteristic function. Clearly, χ_h is bounded and $\chi_h \to 1$ in $L^1(\Omega)$. Thus, we have $\chi_h G_h \rightharpoonup G$ in $L^2(\Omega)$. Expanding $\mathcal{W}$, we get

$$\mathcal{W}(\nabla_h y_h) \geq \mathcal{W}(U_j + h^\gamma G_h) \geq \frac{1}{2} Q_j(h^\gamma G_h) - m(h^\gamma |G_h|) h^\alpha |G_h|^2,$$

where m is the modulus of continuity of $D^2\mathcal{W}$ at U_j. In particular

$$\begin{aligned} E_h^\alpha(y_h) &\geq \frac{1}{2}\int_\Omega \left[Q_j(\chi_h G_h) - m(h^\gamma |\chi_h G_h|)|\chi_h G_h|^2 \right] dx \\ &\geq \frac{1}{2}\int_\Omega Q_j(\chi_h G_h)\, dx - Cm(h^{\gamma-1}). \end{aligned}$$

Recall that Q_j is weakly lower semicontinuous in $L^2(\Omega)$ by convexity. Thus, passing to the limit and applying Lemma 2.2.5 we obtain

$$\begin{aligned} \liminf_{h\to 0} E_h^\alpha(y_h) &\geq \frac{1}{2}\int_\Omega Q_j(G)\, dx = \int_\Omega Q_j(U_j^{-1} \operatorname{sym}(U_j G))\, dx \\ &\geq \int_\Omega \bar{Q}_j(\operatorname{sym}(U_j G)')\, dx. \end{aligned}$$

Recall that

$$\operatorname{sym}(U_j G(x', x_3))' = \operatorname{sym}(U_j G_a(x'))' + x_3 \operatorname{sym}(U_j G_b(x'))'.$$

By Corollary 3.2.9 and Corollary 3.2.13 we conclude. □

3.3.2 Recovery sequences

We are left with the construction of the recovery sequences. Here, for a clearer exposition, we follow the reverse order and start with the recovery sequence for the case $\alpha > 4$.

Proof of Theorem 3.1.3–(iv). Let $j \in \{1, \dots, l\}$. Suppose $\alpha > 4$. By a standard density argument it is sufficient to exhibit a recovery sequence for $u \in C^\infty(\bar{S}; \mathbb{R}^2)$ and $v \in C^\infty(\bar{S})$. We define

$$B_h(x', x_3) := \begin{pmatrix} h^\gamma u \\ h^{\gamma-1} v \end{pmatrix} - h^\gamma x_3 \begin{pmatrix} \nabla' v^T \\ 0 \end{pmatrix} + \frac{h^{\gamma+1}}{2} x_3^2 \xi(x') + h^{\gamma+1} x_3 \zeta(x'),$$

and

$$y_h(x', x_3) := U_j \begin{pmatrix} x' \\ h x_3 \end{pmatrix} + U_j^{-1} B_h(x', x_3),$$

where ξ, ζ are smooth functions independent of x_3 to be chosen later. We immediately deduce that

(i) $h^{1-\gamma} v_h = v + \dfrac{1}{24} h^2 \xi_3 \to v$ in $W^{1,2}(\Omega)$,

(ii) $h^{-\gamma} u_h = u + \dfrac{1}{24} h \xi_{12} \to u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$.

Computing the rescaled gradient, we get

$$\nabla_h y_h = U_j + U_j^{-1} \nabla_h B_h(x', x_3),$$

where

$$\nabla_h B_h = h^\gamma \left[\begin{pmatrix} \nabla' u - x_3 (\nabla')^2 v & -h^{-1} \nabla' v^T \\ h^{-1} \nabla' v & 0 \end{pmatrix} + (x_3 \xi + \zeta) \otimes e_3 \right] + O(h^{\gamma+1}).$$

By construction, we have $\nabla_h y_h = M_h U_j$, where

$$M_h := \mathrm{Id} + U_j^{-1} \nabla_h B_h U_j^{-1}.$$

We first compute $M_h^T M_h$ and obtain

$$M_h^T M_h = \mathrm{Id} + 2 U_j^{-1} \operatorname{sym}(\nabla_h B_h) U_j^{-1} + O(h^{2\gamma-2}). \tag{3.42}$$

Note that $\operatorname{sym}(\nabla_h B_h) = O(h^\gamma)$. Then, we develop $\sqrt{M_h^T M_h}$ near the identity to obtain

$$\sqrt{M_h^T M_h} = \mathrm{Id} + U_j^{-1} \operatorname{sym}(\nabla_h B_h) U_j^{-1} + O(h^{2\gamma-2}).$$

Let $\widetilde{W}(P) = W(P U_j)$ for every $P \in \mathbb{R}^{3\times 3}$. Clearly $\widetilde{W}$ is frame indifferent and

$$D^2 \widetilde{W}(\mathrm{Id}) M : M = Q_j(M U_j).$$

Note that $\gamma - 2 > 0$ whenever $\alpha > 4$. Developing $\widetilde{\mathcal{W}}$ near the identity and recalling Lemma 2.2.8, we obtain

$$\begin{aligned}
\mathcal{W}(\nabla_h y_h) &= \widetilde{\mathcal{W}}(M_h) = \widetilde{\mathcal{W}}\left(\sqrt{M_h^T M_h}\right) \\
&= D^2\widetilde{\mathcal{W}}(\mathrm{Id})(M_h - \mathrm{Id}) : (M_h - \mathrm{Id}) + o(|F - \mathrm{Id}\,|^2) \\
&= h^{2\gamma}\frac{1}{2}Q_j(h^{-\gamma}U_j^{-1}\operatorname{sym}(\nabla_h B_h) + O(h^{\gamma-2})) + o(h^{2\gamma}).
\end{aligned}$$

Observing that $\nabla_h^2 y_h = U_j^{-1}\nabla_h^2 B_h = O(h^{\gamma-1})$, we get by assumption (P2)

$$\lim_{h\to 0^+} E_h^\alpha(y_h) = \frac{1}{2}\int_\Omega Q_j(U_j^{-1}\operatorname{sym}(B))\,dx,$$

where

$$B := U_j^{-1}\left(\begin{pmatrix} \nabla' u - x_3(\nabla')^2 v & 0 \\ 0 & 0\end{pmatrix} + x_3\xi\otimes e_3 + \zeta\otimes e_3\right).$$

Choose $\xi := -2L_j((\nabla')^2 v)$ and $\zeta := 2L_j(\operatorname{sym}(\nabla' u))$ where L_j is the linear operator defined in (2.9). Thus,

$$\begin{aligned}
\frac{1}{2}\int_\Omega Q_j(U_j^{-1}\operatorname{sym}(B))\,dx &= \frac{1}{24}\int_S Q_j(U_j^{-1}((\nabla')^2 v + \operatorname{sym}(\xi\otimes e_3)))\,dx' \\
&\quad + \frac{1}{2}\int_S Q_j(U_j^{-1}(\operatorname{sym}(\nabla' u) + \operatorname{sym}(\zeta\otimes e_3)))\,dx' \\
&= \frac{1}{24}\int_\Omega \bar{Q}_j((\nabla')^2 v)\,dx' + \frac{1}{8}\int_\Omega \bar{Q}_j(\nabla' u^T + \nabla' u)\,dx'
\end{aligned}$$

as desired. The factor $1/24$ is due to the integration of x_3^2 over I, while the mixed term gives no contribution since x_3 has zero mean over I.

Suppose now $\alpha = 4$. We use the same recovery sequence and the same notation as in the case $\alpha > 4$. The main difference is that terms of order $h^{2\gamma-2}$ in (3.42) cannot be neglected. By some simple computation we get

$$M_h^T M_h = \mathrm{Id} + 2h^2 U_j^{-1}(P_h + V)U_j^{-1} + O(h^3),$$

where

$$\begin{aligned}
P_h &:= \begin{pmatrix} \nabla' u - x_3(\nabla')^2 v & -h^{-1}\nabla' v \\ h^{-1}\nabla' v & 0\end{pmatrix} + (x_3\xi + \zeta)\otimes e_3 + \frac{1}{2}|U_j^{-1}e_3|^2\nabla' v\otimes\nabla' v, \\
V &:= -\left(U_j^{-1}e_3\cdot U_j^{-1}\nabla' v^T\right)\operatorname{sym}(\nabla' v\otimes e_3) + \frac{1}{2}\left|U_j^{-1}\nabla' v^T\right|^2 e_3\otimes e_3.
\end{aligned}$$

Hence, developing the square root we obtain

$$\sqrt{M_h^T M_h} = \mathrm{Id} + h^2 U_j^{-1}\operatorname{sym}(P_h + V)U_j^{-1} + O(h^3).$$

Note that $\operatorname{sym}(P_h)$ is independent of h. To conclude it is then sufficient to choose $\xi := -2L_j((\nabla')^2 v)$ and

$$\begin{aligned}
\zeta &:= -\frac{1}{2}\left|U_j^{-1}\nabla' v^T\right|^2 e_3 + \left(U_j^{-1}e_3\cdot U_j^{-1}\nabla' v^T\right)\nabla' v \\
&\quad + 2L_j\left(\operatorname{sym}(\nabla' u) + \frac{1}{2}|U_j^{-1}e_3|^2\nabla' v\otimes\nabla' v\right).
\end{aligned}$$

□

In the case $2 < \alpha < 4$ the construction of the recovery sequence is different and involves the perturbation of an isometric immersion. In doing so, we will need some higher regularity of the boundary of S to deal with the penalty term. We start by proving a preliminary Lemma.

Lemma 3.3.1. *Let $U \in \mathbb{R}^{3\times 3}$ be a symmetric and positive definite matrix. For $\varepsilon > 0$ let $y_\varepsilon \in W^{2,2}(S;\mathbb{R}^3)$ be a map of the form*

$$y_\varepsilon(x') := U\begin{pmatrix} x' \\ 0 \end{pmatrix} + \varepsilon v U^{-1}e_3 + \varepsilon^2 U \begin{pmatrix} u_\varepsilon \\ 0 \end{pmatrix},$$

such that $\nabla' y^T \nabla' y = (U^2)'$, where $v \in W^{2,\infty}(S)$ and u_ε is a bounded sequence in $W^{2,\infty}(S;\mathbb{R}^2)$. Then, if ν_ε is defined as in Lemma 3.2.8, we have

$$\nabla' y_\varepsilon^T \nabla' \nu_\varepsilon = -\varepsilon(\nabla')^2 v + O(\varepsilon^2).$$

Proof. First, we compute $\partial_1 y_\varepsilon \wedge \partial_2 y_\varepsilon$. We have

$$\partial_1 y_\varepsilon \wedge \partial_2 y_\varepsilon = Ue_1 \wedge Ue_2 + \varepsilon\left[Ue_1 \wedge U^{-1}\begin{pmatrix} 0 \\ \partial_2 v\end{pmatrix} + U^{-1}\begin{pmatrix} 0 \\ \partial_1 v\end{pmatrix} \wedge Ue_2\right] + O(\varepsilon^2).$$

Given the identity $Ua \wedge Ub = \det(U)U^{-1}(a \wedge b)$, we easily deduce that

$$\begin{aligned}\partial_1 y_\varepsilon \wedge \partial_2 y_\varepsilon = {} & Ue_1 \wedge Ue_2 \\ & + \varepsilon \det(U)U^{-1}\left[\partial_2 v e_1 \wedge (U^{-1})^2 e_3 + \partial_1 v (U^{-1})^2 e_3 \wedge e_2\right] + O(\varepsilon^2).\end{aligned}$$

Computing the cross products, we get

$$\partial_1 y_\varepsilon \wedge \partial_2 y_\varepsilon = Ue_1 \wedge Ue_2 - \varepsilon \det(U)U^{-1}\begin{pmatrix} |U^{-1}e_3|^2 \partial_1 v \\ |U^{-1}e_3|^2 \partial_2 v \\ -\sum_{k=1}^2 (U^{-1}e_k \cdot U^{-1}e_3)\partial_k v\end{pmatrix} + O(\varepsilon^2).$$

Let us set $q(x') := -\frac{1}{|U^{-1}e_3|^2}\sum_{k=1}^2 (U^{-1}e_k \cdot U^{-1}e_3)\partial_k v$. We have

$$\nabla'(\partial_1 y_\varepsilon \wedge \partial_2 y_\varepsilon) = -\varepsilon |U^{-1}e_3|^2 \det(U)U^{-1}\begin{pmatrix} (\nabla')^2 v \\ \nabla' q\end{pmatrix} + O(\varepsilon^2).$$

Now observe that for every triplet of indices $i, j, k = 1, 2$ we have

$$\partial_i y_\varepsilon \cdot \partial^2_{jk} y_\varepsilon = O(\varepsilon^2).$$

Moreover,

$$\partial_i y_\varepsilon^T U^{-1} = e_i^T + O(\varepsilon).$$

Combining the previous equations with the definition of ν_ε, the thesis follows. □

Proof of Theorem 3.1.3–(iii). Firstly, suppose that $v \in C^\infty(\bar{S})$ satisfies (3.5). By Proposition 3.1.2 we have $\det((\nabla')^2 v) = 0$ in S. For $h \ll 1$ we can apply Theorem 2.2.11 and construct a sequence $u_h \in W^{2,\infty}(S;\mathbb{R}^2)$ with $\|u_h\|_{W^{2,\infty}}$ uniformly bounded such that the map

$$\tilde{y}_h(x') := U_j\begin{pmatrix} x' \\ 0\end{pmatrix} + h^{\gamma-1} v U_j^{-1} e_3 + h^{2\gamma-2} U_j \begin{pmatrix} u_h \\ 0 \end{pmatrix}$$

satisfies $\nabla'\tilde{y}_h^T\nabla'\tilde{y}_h = (U_j^2)'$. Define ν_h as in Lemma 3.2.8 with $U = U_j$ for the map $\tilde{y}_h$. We consider the recovery sequence given by

$$y_h(x', x_3) := \tilde{y}_h(x') + hx_3\nu_h(x') + \frac{1}{2}h^{\gamma+1}x_3^2 U_j^{-1}\xi(x'),$$

where ξ is a smooth function independent of x_3 to be determined later. Observe that

$$\nu_h = v + h^{\gamma-1}U_j^2\begin{pmatrix} u_h \\ 0 \end{pmatrix} \cdot e_3 + \frac{1}{24}h^2\xi_3 \to v \quad \text{in } W^{1,2}(S).$$

We have

$$\nabla_h y_h = \begin{pmatrix} \nabla'\tilde{y}_h & \nu_h \end{pmatrix} + hx_3\begin{pmatrix} \nabla'\nu_h & 0 \end{pmatrix} + h^\gamma x_3 U_j^{-1}\xi \otimes e_3 + O(h^{\gamma+1}). \tag{3.43}$$

Define $R_h := \begin{pmatrix} \nabla'\tilde{y}_h & \nu_h \end{pmatrix} U_j^{-1}$. By definition of ν_h, we have $R_h \in \mathrm{SO}(3)$. We rewrite $\nabla_h y_h$ as

$$\nabla_h y_h = R_h(U_j + B_h),$$

where

$$B_h := hx_3 U_j^{-1}\begin{pmatrix} \nabla'\tilde{y}_h^T\nabla'\nu_h & 0 \\ 0 & 0 \end{pmatrix} + h^\gamma x_3 R_h^T U_j^{-1}\xi \otimes e_3 + O(h^{\gamma+1}).$$

Note that we used the fact that $\nu_h^T\nabla'\nu_h = 0$, which follows differentiating $|\nu_h| \equiv 1$. By Lemma 3.3.1 with $\varepsilon = h^{\gamma-1}$ we have

$$\nabla'\tilde{y}_h^T\nabla'\nu_h = -h^{\gamma-1}(\nabla')^2 v + O(h^{2\gamma-2}).$$

Moreover, it is easy to check that $R_h = \mathrm{Id} + O(h^{\gamma-1})$. Thus,

$$B_h = h^\gamma U_j^{-1}\begin{pmatrix} -x_3(\nabla')^2 v & 0 \\ 0 & 0 \end{pmatrix} + h^\gamma x_3 U_j^{-1}\xi \otimes e_3 + O(h^{2\gamma-1}),$$

and $\nabla_h y_h \to U_j$ in $L^2(\Omega;\mathbb{R}^{3\times3})$. In particular, $h^{-\gamma}B_h \to B$ in $L^2(\Omega;\mathbb{R}^{3\times3})$, where

$$B := U_j^{-1}\begin{pmatrix} -x_3(\nabla')^2 v & 0 \\ 0 & 0 \end{pmatrix} + x_3 U_j^{-1}\xi \otimes e_3.$$

Developing $\mathcal{W}$ we get

$$\mathcal{W}(\nabla_h y_h) = \frac{1}{2}Q_j(B_h) + o(h^{2\gamma}).$$

We are left to estimate the penalty term. We have $\nabla_h^2\tilde{y}_h = O(h^{\gamma-1})$ and, by definition of ν_h, $\nabla'\nu_h = O(h^{\gamma-1})$. By (3.43), it follows that $\nabla_h^2 y_h = O(h^{\gamma-1})$. Hence, by (P2), we deduce that

$$\lim_{h\to 0^+} E_h^\alpha(y_h) = \frac{1}{2}\int_\Omega Q_j(B)\,dx.$$

To conclude, it is sufficient to choose $\xi := -2L_j((\nabla')^2 v)$, where L_j is the linear operator defined in (2.9).

For the general case of $v \in W^{2,2}(S)$ satisfying (3.5) we apply Corollary 2.2.13 and a standard diagonal argument to conclude. □

We are left to construct the recovery sequence for $\alpha = 2$.

Proof of Theorem 3.1.1–(iii). Let $j \in \{1, \dots, l\}$. Firstly, suppose that $y \in C^\infty(\bar{S};\mathbb{R}^3)$ satisfies $(\nabla' y)^T \nabla' y = (U_j^2)'$. Define

$$y_h(x', x_3) := y(x') + hx_3\nu(x') + \frac{1}{2}h^2 x_3^2 \xi(x'),$$

where ξ is a smooth function independent of x_3 to be chosen and ν is defined as in Lemma 3.2.8. Let $R := (\nabla' y \quad \nu)\, U_j^{-1}$. By construction $R \in \mathrm{SO}(3)$ a.e. in S. Computing the rescaled gradient we get

$$\begin{aligned}\nabla_h y_h &= (\nabla' y \quad \nu) + hx_3\,(\nabla'\nu \quad \xi) + \frac{1}{2}h^2 x_3^2\,(\nabla'\xi \quad 0)\\ &= R\left[U_j + hx_3 U_j^{-1}(\nabla' y^T \nabla'\nu) + hx_3 R^T\xi \otimes e_3 + O(h^2)\right].\end{aligned}$$

It is clear that $\nabla_h y_h \to (\nabla' y \quad \nu)$ in $L^2(S;\mathbb{R}^{3\times 3})$. For $h \ll 1$ we expand $\mathcal{W}$ and get

$$\mathcal{W}(\nabla_h y_h) = \frac{1}{2}h^2 x_3^2 Q_j\left(U_j^{-1}(\nabla' y^T \nabla'\nu) + R^T\xi \otimes e_3\right) + o(h^2).$$

By the symmetry of Q_j (see Lemma 2.2.5) we immediately deduce that

$$\mathcal{W}(\nabla_h y_h) = \frac{1}{2}h^2 x_3^2 Q_j\left(U_j^{-1}(\nabla' y^T \nabla'\nu + \mathrm{sym}(U_j R^T\xi \otimes e_3))\right) + o(h^2).$$

Indeed, since $\partial_j(\nabla' y^T \nu) = 0$, the matrix $\nabla' y^T \nabla'\nu$ is symmetric by the following chain of equalities

$$\partial_i y \cdot \partial_j \nu = -\partial_{ij} y \cdot \nu = -\partial_{ji} y \cdot \nu = -\partial_j y \cdot \partial_i \nu. \tag{3.44}$$

Set $\xi := RU_j^{-1} L_j(\nabla' y^T \nabla'\nu)$, where L_j is the linear operator defined in (2.9). By construction, we have

$$\frac{1}{h^2}\int_\Omega \mathcal{W}(\nabla_h y_h)\,dx = \frac{1}{24}\int_S \bar{Q}_j(\nabla' y^T \nabla'\nu)\,dx + o(1).$$

Clearly, the rescaled Hessian $\nabla_h^2 y_h$ is bounded in $L^\infty(\Omega;\mathbb{R}^{3\times 3\times 3})$. Hence, by (P2) we have

$$\frac{\eta^p(h)}{h^2}\int_\Omega |\nabla_h^2 y_h|^p\,dx \to 0,$$

concluding the proof of the existence of a recovery sequence for a smooth y.

To conclude the proof, we first observe that E_j^{K} is continuous with respect to the $W^{2,2}$ topology. Let $\mathcal{G}_j := (U_j^2)'$. For every $y \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$, arguing as in (3.44), we have $(\nabla' y)^T \nabla'\nu = -(\nabla')^2 y\nu$, where ν is defined as in Lemma 3.2.8. Given a sequence $(y_n) \subset W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$ such that $y_n \to y$ in $W^{2,2}(S;\mathbb{R}^3)$ we have, up to a subsequence, $(\nabla')^2 y_n \to (\nabla')^2 y$ almost everywhere in S. Let ν_n and ν be defined as in Lemma 3.2.8 for y_n and y, respectively. Then, $\nu_n \to \nu$ in $L^1(S;\mathbb{R}^3)$, thus, up to subsequences, $\nu_n \to \nu$ almost everywhere in S. Hence, $(\nabla')^2 y_n \nu_n \to (\nabla')^2 y\nu$ and by Dominated Convergence Theorem $E_j^{\mathrm{K}}(y_n) \to E_j^{\mathrm{K}}(y)$.

Moreover, the set of functions $y \in C^\infty(\bar{S};\mathbb{R}^3)$ satisfying $(\nabla' y)^T \nabla' y = (U_j^2)'$ is dense in $W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$. Indeed, let $\varepsilon > 0$ and pick an isometric immersion $y \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$. Define $\tilde{y}(x) := y(\mathcal{G}_j^{-1/2}x)$. Clearly, $\tilde{y} \in W^{2,2}(\mathcal{G}^{1/2}S;\mathbb{R}^3)$ and

$$\nabla' \tilde{y}^T \nabla' \tilde{y} = \mathcal{G}_j^{-1/2} \nabla' y^T \nabla' y \mathcal{G}_j^{-1/2} = \mathrm{Id}\,.$$

The set $\mathcal{G}_j^{1/2}S$ satisfies condition (2.17). Hence, by Theorem 2.2.12, there exists a smooth isometric immersion $\tilde{\phi}$ for the flat metric Id such that $\|\tilde{y} - \tilde{\phi}\|_{W^{2,2}} \leq \varepsilon\sqrt{\det(G_j^{1/2})}$. Defining $\phi(x) := \tilde{\phi}(\mathcal{G}_j^{1/2}x)$ we get $\phi \in C^\infty(\bar{S};\mathbb{R}^3)$,

$$\nabla\phi^T \nabla\phi = \mathcal{G}_j = (U_j^2)',$$

and $\|y - \phi\|_{W^{2,2}} \leq \varepsilon$. A standard diagonal argument allows us to conclude. $\square$

Remark 3.3.2. Observe that the argument used in [FJM02] to prove the existence of a recovery sequence cannot be applied here. Indeed, the truncation argument, which is the basis of the construction of [FJM02], would lead to deformations with a low regularity, for which the penalty term cannot be written.

3.4 Convergence of minimizers with dead loads

In this section we prove Theorem 3.1.6 and Theorem 3.1.8. We start by showing that a sequence of deformations with bounded total energy has also bounded elastic energy.

Lemma 3.4.1. *Let $\alpha \geq 2$ and $q > 1$. Suppose that $(y_h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ is a sequence of deformations that are quasi-minimizers for J_h^α, that is,*

$$\lim_{h\to 0} (J_h^\alpha(y_h) - \inf J_h^\alpha) = 0.$$

Then, $E_h^\alpha(y_h) \leq C$ for every $h > 0$.

Proof. Firstly, we will prove that $h^\alpha E_h^\alpha(y_h) \to 0$. Fix $j \in \{1,\ldots,l\}$ and let Q_h, R_h be, respectively, the rotation and the map given by Proposition 3.2.6. Define

$$\tilde{y}_h := y_h - Q_h U_j \begin{pmatrix} x' \\ hx_3 \end{pmatrix} - c_h,$$

where

$$c_h := \frac{1}{|\Omega|}\int_\Omega \left(y_h - Q_h U_j \begin{pmatrix} x' \\ hx_3 \end{pmatrix}\right) dx.$$

Using the test deformation

$$\begin{pmatrix} x' \\ x_3 \end{pmatrix} \mapsto U_j \begin{pmatrix} x' \\ hx_3 \end{pmatrix}$$

and the assumption (3.10), we get $\inf_y J_h^\alpha(y) \leq Ch^{1-\gamma}$, so that

$$J_h^\alpha(y_h) - Ch^{1-\gamma} \leq J_h^\alpha(y_h) - \inf_y J_h^\alpha(y) = o(1) \implies J_h^\alpha(y_h) \leq Ch^{1-\gamma}.$$

We have

$$
\begin{aligned}
h^\alpha E_h^\alpha(y_h) &= h^\alpha J_h^\alpha(y_h) + \int_\Omega f_h \cdot \tilde{y}_h \, dx + \int_S f_h \cdot Q_h U_j \begin{pmatrix} x' \\ 0 \end{pmatrix} dx \\
&\leq C h^{\gamma+1} + C h^{\gamma+1} \|\nabla_h \tilde{y}_h\|_{L^q}.
\end{aligned}
\tag{3.45}
$$

Consider the set

$$
\tilde{\Omega}_h := \{x \in \Omega \colon \operatorname{dist}(\nabla_h y_h, K) \geq 1\}.
$$

By Proposition 3.2.6 and Remark 3.2.7, we deduce

$$
\begin{aligned}
\|\nabla_h \tilde{y}_h\|_{L^q}^q &= \int_\Omega |\nabla_h y_h - Q_h U_j|^q \, dx \leq C h^{-2} \int_\Omega \operatorname{dist}^q(\nabla_h y_h, K^j) \, dx \\
&\leq C h^{-2} \int_\Omega \operatorname{dist}^q(\nabla_h y_h, K) \, dx + C h^{-2} \\
&\leq C h^{-2} \int_{\tilde{\Omega}_h} \operatorname{dist}^q(\nabla_h y_h, K) \, dx + C h^{-2} \\
&\leq C h^{\alpha-2} E_h^\alpha(y_h) + C h^{-2},
\end{aligned}
\tag{3.46}
$$

where we have used (3.1). Combining (3.45)–(3.46) we get

$$
h^\alpha E_h^\alpha(y_h) \leq C h^{\gamma+1} + C h^{\gamma+1-\frac{2}{q}} (h^\alpha E_h^\alpha(y_h))^{\frac{1}{q}} + C h^{\gamma+1-\frac{2}{q}}.
$$

Recalling that $q > 1$, by Young's inequality we deduce that

$$
h^\alpha E_h^\alpha(y_h) \leq C h^{\gamma+1} + C h^{q'(\gamma+1-\frac{2}{q})} + C h^{\gamma+1-\frac{2}{q}} \to 0.
$$

Hence, we can apply Proposition 3.2.1 and deduce that, at least along a subsequence, there is an index $j_0 \in \{1, \ldots, l\}$ such that for $h \ll 1$

$$
\int_\Omega \operatorname{dist}^2(\nabla_h y_h, K^{j_0}) \, dx \leq h^\alpha [E_h^\alpha(y_h) + (E_h^\alpha(y_h))^\theta].
$$

We now show that $E_h^\alpha(y_h) \leq C$. To simplify the exposition, we suppose that $j_0 = j$. By Proposition 3.2.6 we have

$$
\begin{aligned}
\|\tilde{y}_h\|_{L^2}^2 \leq C \|\nabla \tilde{y}_h\|_{L^2}^2 &\leq C h^{-2} \|\operatorname{dist}(\nabla_h y_h, K^j)\|_{L^2}^2 \\
&\leq C h^{\alpha-2} [E_h^\alpha(y_h) + (E_h^\alpha(y_h))^\theta].
\end{aligned}
\tag{3.47}
$$

Pick $\bar{R}_h U_{k_h} \in \mathcal{M}_h$ and define

$$
\tilde{J}_h^\alpha(y) := E_h^\alpha(y) - \frac{1}{h^\alpha} \int_\Omega f_h \cdot y \, dx + \frac{1}{h^\alpha} \int_S f_h \cdot \bar{R}_h U_{k_h} \begin{pmatrix} x' \\ 0 \end{pmatrix} dx.
$$

Note that y_h are quasi-minimizers for the functional $\tilde{J}_h^\alpha$. Moreover, using the test deformation

$$
\begin{pmatrix} x' \\ x_3 \end{pmatrix} \mapsto \bar{R}_h U_{k_h} \begin{pmatrix} x' \\ h x_3 \end{pmatrix},
$$

we get $\inf_y \tilde{J}_h^\alpha(y) \leq 0$, so that

$$
\tilde{J}_h^\alpha(y_h) \leq \tilde{J}_h^\alpha(y_h) - \inf_y \tilde{J}_h^\alpha(y) \leq C.
$$

Hence,

$$
\begin{aligned}
E_h^\alpha(y_h) &= \tilde{J}_h^\alpha(y_h) + \frac{1}{h^\alpha}\int_\Omega f_h \cdot y_h \, dx - \frac{1}{h^\alpha}\int_S f_h \cdot \bar{R}_h U_{k_h} \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\
&= \tilde{J}_h^\alpha(y_h) + \frac{1}{h^\alpha}\int_\Omega f_h \cdot \tilde{y}_h \, dx + \frac{1}{h^\alpha}\int_S f_h \cdot (Q_h U_j - \bar{R}_h U_{k_h}) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\
&\leq C + C h^{-\gamma-1} \|f_h\|_{L^2} (E_h^\alpha(y_h) + (E_h^\alpha(y_h))^\theta)^{\frac{1}{2}} \\
&\leq C + C(E_h^\alpha(y_h))^{\frac{1}{2}} + C(E_h^\alpha(y_h))^{\frac{\theta}{2}},
\end{aligned}
$$

where we have used (3.47) and the definition of $\mathcal{M}_h$. Since $0 < \theta/2 < 1$, by an application of Young's inequality we conclude. □

We are in a position to conclude the proof of Theorem 3.1.6.

Proof of Theorem 3.1.6. By Lemma 3.4.1 and Theorem 3.1.1–(i) there exists $j \in \{1, \ldots, l\}$ and $y \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$ such that, up to a subsequence, $(\nabla_h y_h) \to (\nabla' y \quad \nu)$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$, where ν is defined as in Lemma 3.2.8. Let

$$
c_h := \frac{1}{|\Omega|}\int_\Omega y_h \, dx
$$

and define $\tilde{y}_h := y_h - c_h$. By an application of the Poincaré–Wirtinger inequality, we deduce that $\tilde{y}_h \to y$ in $W^{1,2}(S;\mathbb{R}^3)$. By the strong convergence of $h^{-2} f_h$ and (3.11), we get

$$
\frac{1}{h^2}\int f_h \cdot y_h \, dx = \frac{1}{h^2}\int f_h \cdot \tilde{y}_h \, dx \to \int_S f \cdot y \, dx.
$$

By Theorem 3.1.1–(ii) we deduce that

$$
\liminf_{h\to 0} J_h^2(y_h) \geq E_j^{\mathrm{K}}(y) - \int_S f \cdot y \, dx = J_j^{\mathrm{K}}(y).
$$

Now let $i \in \{1, \ldots, l\}$ and $\bar{y} \in W^{2,2}_{\mathrm{iso},\mathcal{G}_j}(S;\mathbb{R}^3)$. Let $(\bar{y}_h)$ be the recovery sequence for $\bar{y}$ provided by Theorem 3.1.1–(iii). Then

$$
\begin{aligned}
J_j^{\mathrm{K}}(y) &\leq \liminf_{h\to 0} J_h^2(y_h) \leq \limsup_{h\to 0} J_h^2(y_h) = \limsup_{h\to 0} (\inf_y J_h^2(y)) \\
&\leq \limsup_{h\to 0} J_h^2(\bar{y}_h) = \lim_{h\to 0} J_h^2(\bar{y}_h) = J_i^{\mathrm{K}}(\bar{y}),
\end{aligned}
$$

concluding the proof. □

We move now to the case $\alpha > 2$ proving Theorem 3.1.8.

Proof of Theorem 3.1.8. By Lemma 3.4.1, we have $E_h^\alpha(y_h) \leq C$. Let $\bar{R}_h$, u_h, v_h and $j \in \{1, \ldots, l\}$ be the sequences and the index given by Theorem 3.1.3–(i). Up to subsequences, we have $\bar{R}_h \to \bar{R}$ in $\mathrm{SO}(3)$. We prove that $\bar{R}U_j \in \mathcal{M}$. This will also show that $j \in \Lambda$. Indeed, let $RU_k \in K$ and define

$$
\bar{y}_h(x', x_3) := RU_k \begin{pmatrix} x' \\ h x_3 \end{pmatrix}.
$$

We get

$$
\begin{aligned}
J_h^\alpha(y_h) - \inf_y J_h^\alpha(y) &\geq J_h^\alpha(y_h) - J_h^\alpha(\bar{y}_h) \\
&\geq -\frac{1}{h^\alpha}\int_\Omega f_h \cdot y_h \, dx + \frac{1}{h^\alpha}\int_\Omega f_h \cdot RU_k \begin{pmatrix} x' \\ 0 \end{pmatrix} dx \\
&= -\frac{1}{h^\alpha}\int_\Omega f_h \cdot \bar{R}_h U_j^{-1} \begin{pmatrix} \max\{h^\gamma, h^{2\gamma-2}\}u_h \\ h^{\gamma-1}v_h \end{pmatrix} dx \\
&\quad + \frac{1}{h^\alpha}\int_\Omega f_h \cdot (RU_k - \bar{R}_h U_j) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx.
\end{aligned} \tag{3.48}
$$

Multiplying by $h^{\gamma-1}$ and passing to the limit we deduce that

$$
\int_\Omega f \cdot (RU_k - \bar{R}U_j) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx \leq 0,
$$

where we have used convergence (3.10). Since $RU_k \in K$ is arbitrary, we have $\bar{R}U_j \in \mathcal{M}$.

By Proposition 2.2.20, the projection $P_h^j(R_h)$ is well-defined for $h \ll 1$. Let $W_h \in \mathbb{R}^{3\times 3}_{\text{skew}}$ with $|W_h| = 1$ be such that

$$
P_h^j(\bar{R}_h)W_h \in N\mathcal{R}^j_{h\,P_h^j(\bar{R}_h)}
$$

and $\bar{R}_h := P_h^j(\bar{R}_h)e^{d_h W_h}$, where $d_h := \text{dist}_{SO(3)}(\bar{R}_h, P_h^j(\bar{R}_h))$. Then, by (3.48) with $RU_k = P_h^j(\bar{R}_h)U_j$ and the fact that $F_h(P_h^j(\bar{R}_h)W_hU_j) = 0$ (see Section 2.2.3) we have, expanding the exponential map,

$$
\begin{aligned}
J_h^\alpha(y_h) - \inf_y J_h^\alpha(y) &\geq J_h^\alpha(y_h) - J_h^\alpha\left(P_h^j(\bar{R}_h)U_j \begin{pmatrix} x' \\ hx_3 \end{pmatrix}\right) \\
&\geq -C - \frac{d_h^2}{h^\alpha}\int_\Omega f_h \cdot P_h^j(\bar{R}_h)W_h^2 U_j \, dx + O\left(\frac{d_h^3}{h^{\gamma-1}}\right).
\end{aligned} \tag{3.49}
$$

Clearly $P_h^j(\bar{R}_h) \to \bar{R}$. Moreover, up to subsequences, $W_h \to W$ and owing to Lemma 2.2.17, $\bar{R}W \in N\mathcal{R}^j_{\bar{R}}$, thus $F(\bar{R}W^2U_j) < 0$. We deduce by (3.49) that $d_h^2 = O(h^{\gamma-1})$. In particular, there exists $\beta \geq 0$ such that $h^{1-\gamma}d_h^2 \to \beta^2$ and so

$$
h^{\frac{1}{2}(1-\gamma)}(\bar{R}_h - P_h^j(\bar{R}_h)) = \frac{d_h}{h^{\frac{1}{2}(\gamma-1)}}P_h^j(\bar{R}_h)W_h + O\left(\frac{d_h^2}{h^{\frac{1}{2}(\gamma-1)}}\right) \to \beta\bar{R}W.
$$

We are left to prove the minimality property. We show it for $2 < \alpha < 4$. The other cases can be proved analogously. Firstly, note that

$$
\frac{1}{h^\alpha}\int_\Omega f_h \cdot y_h \, dx - \frac{1}{h^\alpha}F_h(P_h^j(\bar{R}_h)U_j) \to \int_S f \cdot \bar{R}U_j^{-1}e_3 v \, dx - F(\beta^2\bar{R}W^2U_j).
$$

Indeed, we have the equality

$$
\begin{aligned}
\frac{1}{h^\alpha}\int_\Omega f_h \cdot y_h \, dx - F(P_h^j(\bar{R}_h)U_j) &= \frac{1}{h^\alpha}\int_\Omega f_h \cdot \bar{R}_h U_j^{-1} \begin{pmatrix} h^{2\gamma-2}u_h \\ h^{\gamma-1}v_h \end{pmatrix} dx \\
&\quad - \frac{1}{h^\alpha}\int_\Omega f_h \cdot (P_h^j(\bar{R}_h) - \bar{R}_h)U_j \begin{pmatrix} x' \\ 0 \end{pmatrix} dx.
\end{aligned}
$$

The first term behaves as follows

$$\frac{1}{h^\alpha}\int_\Omega f_h\cdot \bar R_h U_j^{-1}\begin{pmatrix} h^{2\gamma-2}u_h\\ h^{\gamma-1}v_h\end{pmatrix}\,dx = \int_\Omega \frac{1}{h^{\gamma+1}} f_h\cdot \bar R_h U_j^{-1}\begin{pmatrix} h^{\gamma-1}u_h\\ v_h\end{pmatrix}\,dx \\ \to \int_S f\cdot \bar R U_j^{-1}e_3 v\,dx.$$

On the other hand, since $F_h(P_h^j(\bar R_h)W_hU_j)=0$, we have

$$\frac{1}{h^\alpha}F_h(P_h^j(\bar R_h)U_j) = \frac{d_h^2}{h^{\gamma-1}}\int_\Omega \frac{1}{h^{\gamma+1}} f_h\cdot P_h^j(\bar R_h)W_h^2U_j\begin{pmatrix} x'\\ 0\end{pmatrix}\,dx + O\left(\frac{d_h^3}{h^{\gamma-1}}\right),$$

that converges to $F(\beta^2\bar R W^2U_j)$. Thus, by Theorem 3.1.3–(ii) we have

$$\liminf_{h\to 0}(J_h^\alpha(y_h)+F(P_h^j(\bar R_h)U_j)) \ge J_j^{CVK}(v,\bar R,\beta W). \tag{3.50}$$

Take an admissible quadruplet $(i,\tilde v,\tilde R,\tilde W)$ and let $\tilde y_h$ be the recovery sequence for $\tilde v$ provided by Theorem 3.1.3–(iii). By Proposition 2.2.19, we can construct a sequence of rotations $\tilde R_h$ converging to $\tilde R$ such that $\tilde R_h\in\mathcal{R}_h^i$. Note that, by (FI),

$$E_h^\alpha(\tilde y_h) = E_h^\alpha(\tilde R_h\tilde y_h),$$

so that $E_h^\alpha(\tilde R_h\tilde y_h)\to E_i^{CVK}(\tilde v)$. Moreover,

$$\frac{1}{h^\alpha}\int_\Omega f_h\cdot\tilde R_h\tilde y_h - \frac{1}{h^\alpha}F_h(\tilde R_hU_i) = \frac{1}{h^\alpha}\int_S f_h\cdot \tilde R_h U_i^{-1}\begin{pmatrix} h^{2\gamma-2}\tilde u_h\\ h^{\gamma-1}\tilde v_h\end{pmatrix}\,dx \\ \to \int_S f\cdot\tilde R U_i^{-1}e_3\tilde v\,dx,$$

where $\tilde u_h,\tilde v_h$ are defined as in (3.3)–(3.4) with $\tilde y_h$ and U_i in place of y_h and U_j, respectively. By hypothesis (F^1), we have that $\tilde R_hU_i\in\mathcal{M}_h$ for $h\ll 1$. Thus,

$$F_h(P_h^j(\bar R_h)U_j) = F_h(\tilde R_hU_i),$$

and $F(\tilde R(\tilde W)^2U_i)\le 0$. Hence, by (3.50) we deduce that

$$\begin{aligned} J_j^{CVK}(v,\bar R^T,\beta W) &\le \liminf_{h\to 0}\left(J_h^\alpha(y_h)+\frac{1}{h^\alpha}F_h(P_h^j(\bar R_h)U_j)\right)\\ &\le \limsup_{h\to 0}\left(J_h^\alpha(y_h)+\frac{1}{h^\alpha}F_h(P_h^j(\bar R_h)U_j)\right)\\ &= \limsup_{h\to 0}\left(\inf_y J_h^\alpha(y)+\frac{1}{h^\alpha}F_h(P_h^j(\bar R_h)U_j)\right)\\ &\le \limsup_{h\to 0}\left(J_h^\alpha(\tilde R_h\tilde y_h)+\frac{1}{h^\alpha}F_h(\tilde R_hU_i)\right)\\ &= J_i^{CVK}(\tilde v,\tilde R,0)\le J_i^{CVK}(v,\tilde R,\tilde W), \end{aligned}$$

that gives the minimality property. □

4

Stability of the Von Kármán's regime under Neumann boundary conditions

4.1 Assumptions and main results

As in the previous chapter we assume $S \subset \mathbb{R}^2$ to be an open, bounded, and connected set with Lipschitz boundary, representing the mid-plane of a plate. The elastic energy density $\mathcal{W}$ is defined on $\mathbb{R}^{3\times 3}$ and has a single-well structure. Thus, we use the notations presented in Section 2.1, dropping whenever possible the dependence on the well. Recall that in the single-well case we have $l := 1$ and $U_1 := \mathrm{Id}$. We assume that $\mathcal{W}$ satisfies the quadratic growth condition

$$\mathcal{W}(M) \geq C\,\mathrm{dist}^2(M, SO(3)) \qquad \forall\, M \in \mathbb{R}^{3\times 3}.$$

In particular, (2.8) holds.

We assume the applied forces to be of the form

$$f_h := h^2 f, \tag{4.1}$$

with $f \in L^2(S; \mathbb{R}^3)$, f not identically equal to 0.

The total energy for a deformation $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ is written as

$$J_h(y) := I_h(y) - \int_\Omega f_h \cdot y\, dx = \int_\Omega \mathcal{W}(\nabla_h y)\, dx - h^2 \int_\Omega f \cdot y\, dx.$$

We suppose that

$$\int_S f\, dx' = 0 \tag{4.2}$$

to avoid the trivial case in which the total energy has no lower bound.

In this chapter, we use E^{K}, E^{CVK}, and E^{VK} to denote the same functionals we have defined in Chapter 3. However, since there is a single well, we drop the subscript denoting it. Moreover, we repeatedly use the compactness and Γ-convergence results of Chapter 3 replacing E_h^α with $D_h^{-1} I_h$, where $D_h \subset \mathbb{R}^+$ is an infinitesimal sequence such that $D_h = O(h^2)$. Indeed, as we already observed in Remark 3.1.5, these results hold in the single-well case also without penalty term.

Finally, we define the total energy in the Von Kármán's and Kirchhoff's regimes, respectively, as

$$
\begin{aligned}
J^{\mathrm{VK}}(u,v,R,W) &:= E^{\mathrm{VK}}(u,v) - \int_S f \cdot R \begin{pmatrix} u \\ 0 \end{pmatrix} dx' - \int_S f \cdot RW \begin{pmatrix} 0 \\ v \end{pmatrix} dx' \\
&\quad - \int_S f \cdot RW^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx', \\
J^{\mathrm{K}}(y) &:= E^{\mathrm{K}}(y) - \int_S f \cdot y \, dx'.
\end{aligned}
$$

The first functional is defined for every $(u,v) \in W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S)$, $R \in \mathcal{R}$ and $RW \in \mathrm{N}\mathcal{R}_R$ (see Section 2.1 for a definition of $\mathcal{R}$ and $\mathrm{N}\mathcal{R}_R$). A quadruplet (u,v,R,W) as above is called admissible. The Kirchhoff's functional is defined for every $y \in W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$.

Note that, differently from Chapter 3, in the total energy we have a contribution of the in-plane displacement u. This stems from the fact that we are choosing a different scaling for the loads, precisely h^2 in place of h^3.

These energies can be interpreted as the Γ-limit of the corresponding rescalings of J_h. However, the Γ-limit result alone is not satisfactory, since we lack the corresponding compactness properties for sequences with bounded total rescaled energy.

Similarly to the Dirichlet case treated in [LM09], an exclusion principle involving the stability of J^{VK} and J^{K} can be used to study the limit of minimizing sequences in the Von Kármán's regime. In our setting, these stability conditions read as follows:

(S1) $J^{\mathrm{K}}(y) \geq 0$ for every $y \in W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$ and, if $J^{\mathrm{K}}(y) = 0$, then $y = \hat{R} \begin{pmatrix} x' \\ 0 \end{pmatrix}$ for some $\hat{R} \in \mathrm{SO}(3)$,

(S2) $J^{\mathrm{VK}}(u,v,R,W) \geq 0$ for every admissible quadruplet (u,v,R,W) with $(u,v) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ and, if $J^{\mathrm{VK}}(u,v,R,W) = 0$ for some $(u,v) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$, then v is affine,

where

$$
\mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}} := \Big\{ (u,v) \in W^{1,2}(S;\mathbb{R}^2) \times W^{2,2}(S) : \nabla u^T + \nabla u + \nabla v \otimes \nabla v = 0 \text{ a.e.} \Big\}.
$$

Conditions (S1)–(S2) have to be interpreted as follows: whenever a deformation minimizes the (non-negative) total energy then it must be a deformation with zero elastic energy. In our framework, if (S1) holds, then the following compatibility condition is in force:

$$
R^T f \cdot e_3 = 0, \qquad \forall\, R \in \mathcal{R}. \tag{C}
$$

This is the main statement of Theorem 4.1.2. Compatibility condition (C) is the rotation invariant generalization of the usual assumption on the scaling of the normal component of the forces, see for example [FJM06]. Indeed, the standard requirement $f_h \cdot e_3 = O(h^3)$ in our setting translates to $f \cdot e_3 = 0$ (see (4.1)). As illustrated in the following example, condition (C) can be satisfied by very simple loads, despite seeming very restrictive.

Example 4.1.1. Consider $S := (-1/2, 1/2)^2$ and $f := x_1 e_3$. A quick computation gives

$$F(R) = \frac{1}{12} R_{31},$$

thus, $\mathcal{R} = \{R \in \mathrm{SO}(3) \colon R_{31} = 1\}$. In particular for any optimal rotation $R \in \mathcal{R}$ we have $R^T e_3 = e_1$, so that $R^T f \cdot e_3 = (x_1 - 1/2) e_1 \cdot e_3 = 0$.

From now on, unless otherwise stated, $(y_h) \subset W^{1,2}(S; \mathbb{R}^3)$ denotes a quasi-minimizing sequence for $h^{-4} J_h$, namely

$$\limsup_{h \to 0} \frac{1}{h^4} \left(J_h(y_h) - \inf_y J_h(y) \right) = 0. \tag{4.3}$$

Theorem 4.1.2. *Assume that (C) is not valid. Then (S1) fails. Moreover, up to a subsequence, every sequence (y_h) of quasi-minimizers in the sense of (4.3) converge strongly in $W^{1,2}(\Omega; \mathbb{R}^3)$ to a minimizer $\bar{y} \in W^{2,2}_{\mathrm{iso}}(S; R^3)$ of J^{K} with $\bar{y} \neq R \begin{pmatrix} x' \\ 0 \end{pmatrix}$ for every $R \in \mathrm{SO}(3)$.*

Theorem 4.1.2 shows that in the purely Neumann case, some forces are incompatible with the Von Kármán's regime. In particular, if (C) is not in force, the energy of any sequence of quasi-minimizers as in (4.3) scales like h^2, namely

$$0 < \liminf_{h \to 0} \frac{1}{h^2} E_h(y_h) < +\infty.$$

Next, we state the stability alternative analogue to [LM09, Theorem 4].

Theorem 4.1.3. *Let $(y_h) \subset W^{1,2}(S; \mathbb{R}^3)$ be a sequence of quasi-minimizers in the sense of (4.3). Suppose that conditions (S1)–(S2) hold true. Then $\limsup_{h \to 0} h^{-4} E_h(y_h) \leq C$ and there are sequences $(\bar{R}_h) \subset \mathrm{SO}(3)$ and $(c_h) \subset \mathbb{R}^3$ such that, setting*

$$\tilde{y}_h := \bar{R}_h^T y_h + c_h,$$

we have the following convergences (up to a subsequence):

(a) $u_h := \dfrac{1}{h^2} \displaystyle\int_I \left(\tilde{y}_h' - x' \right) \, dx_3 \rightharpoonup \bar{u}$ *in* $W^{1,2}(S; \mathbb{R}^2)$,

(b) $v_h := \dfrac{1}{h} \displaystyle\int_I \tilde{y}_{h,3} \, dx_3 \to \bar{v}$ *in* $W^{1,2}(S)$ *with* $\bar{v} \in W^{2,2}(S)$,

(c) $\bar{R}_h \to \bar{R} \in \mathcal{R}$,

(d) $h^{-1}(P(\bar{R}_h) - \bar{R}_h) \to \bar{R}\bar{W}$ *with* $\bar{R}\bar{W} \in \mathrm{N}\mathcal{R}_{\bar{R}}$, *where* $P : \mathrm{SO}(3) \to \mathcal{R}$ *is the projection onto* $\mathcal{R}$*. Finally, the quadruplet* $(\bar{u}, \bar{v}, \bar{R}, \bar{W})$ *minimizes* J^{VK}*.*

Similarly to [LM09, Theorem 6] we can show that (S1) and (S2) are in a relationship, with the former being essentially stronger than the latter.

Theorem 4.1.4. *Suppose that (S1) holds. Then* $J^{\mathrm{VK}}(u, v, R, W) \geq 0$ *for every admissible quadruplet with* $(u, v) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$*. Moreover, (S2) holds for the functional*

$$J^{\mathrm{VK}}_{\varepsilon}(u, v, R, W) := E^{\mathrm{VK}}(u, v) - (1-\varepsilon) \int_S f \cdot R \begin{pmatrix} u \\ 0 \end{pmatrix} dx' \\ - (1-\varepsilon) \int_S f \cdot RW \begin{pmatrix} 0 \\ v \end{pmatrix} dx' - (1-\varepsilon) \int_S f \cdot RW^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx'$$

for every $\varepsilon \in (0, 1)$*.*

In general the previous result does not hold when $\varepsilon = 0$. Indeed, one can only deduce the positivity of J^{VK} but not the triviality of the minimizers.

The stability conditions are deeply linked to the attainment of the infimum of J^{VK}. Indeed, we prove the following.

Theorem 4.1.5. *Suppose that the stability condition (S2) and the compatibility condition (C) hold true, and that* $\dim \mathcal{R} = 1$*. Then* J^{VK} *attains its minimum over all admissible quadruplets* (u, v, R, W)*. Instead, if (S2) fails, then for every* $\varepsilon > 0$ *the infimum of the functional*

$$J^{\mathrm{VK}}_{\varepsilon}(u, v, R, W) := E^{\mathrm{VK}}(u, v) - (1+\varepsilon) \int_S f \cdot R \begin{pmatrix} u \\ 0 \end{pmatrix} dx' \\ - (1+\varepsilon) \int_S f \cdot RW \begin{pmatrix} 0 \\ v \end{pmatrix} dx' - (1+\varepsilon) \int_S f \cdot RW^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx'$$

is $-\infty$*.*

As for Theorem 4.1.4, also Theorem 4.1.5 might not hold for $\varepsilon = 0$ (see also Remark 4.3.5). Roughly speaking, this result means that the load f is critical, i.e., as soon as the load increases the Von Kármán's model ceases to be valid.

Remark 4.1.6. In Lemma 2.2.22 and Remark 2.2.23 it is proved that the dimension of $\mathcal{R}$ is either zero or one. Theorem 4.1.5 holds also in the case $\dim \mathcal{R} = 0$. However, if $\mathcal{R}$ is a singleton, our setting reduces to the one in [LM09]. For this reason, we only give a sketch of the proof for the case $\dim \mathcal{R} = 0$ (see Remark 4.3.6).

To prove Theorem 4.1.5 a careful analysis of the invariance of J^{VK} along affine perturbations is needed.

4.2 Stability alternative

The aim of this section is to prove Theorem 4.1.2–4.1.4.

In our arguments, we often compare the quasi-minimizing sequence y_h (in the sense of (4.3)) with carefully chosen test deformations $\hat{y}_h$. Indeed, we have

$$J_h(y_h) - J_h(\hat{y}_h) = \inf_y J_h(y) - J_h(\hat{y}_h) + o(h^4) = o(h^4). \tag{4.4}$$

Passing to the limit in (4.4) we deduce relevant properties of the quasi-minimizing sequence y_h. The test deformations $\hat{y}_h$ we use are inspired by the recovery sequence construction of [FJM06] and are similar in spirit to the recovery sequence we used in Chapter 3. For this reason, we refer to Chapter 3 for the explicit computation of their elastic energy.

In order to prove Theorem 4.1.2 it is crucial to have at our disposal the following result, relating the energy scaling of y_h and the compatibility condition (C). Here, and in the rest of the section, we denote by $(D_h) \subset \mathbb{R}^+$ an infinitesimal sequence.

Theorem 4.2.1. *Suppose that* $\limsup_{h\to 0} D_h^{-1} E_h(y_h) \leq C$ *with* $D_h/h^2 \to 0$. *Then* (C) *is in force, i.e.,* $R^T f \cdot e_3 = 0$ *for every* $R \in \mathcal{R}$.

For a quasi-minimizing sequence as in (4.3) we have that $E_h(y_h) \leq Ch^2$ (see the proof of Theorem 4.1.2). Thus, Theorem 4.2.1 equivalently ensures that the elastic energy of y_h scales like h^2 when (C) does not hold true.

To prove Theorem 4.2.1 we proceed by steps, one for each possible limit of D_h/h^4. Every case corresponds to an elastic energy regime. In each step we compare the quasi-minimizing sequence y_h with test deformations having the same elastic energy scaling.

The first part of the section is thus devoted to the proof of Theorem 4.2.1. Once Theorem 4.2.1 is established, we move to the proof of Theorems 4.1.2 to 4.1.4.

We start by proving that, if we are in the Von Kármán's energy scaling, the quasi-minimizing sequence y_h converges, up to a subsequence, to a rigid motion given by an optimal rotation.

Lemma 4.2.2. *Suppose that* $E_h(y_h) \leq CD_h$ *and* $D_h/h^2 \to 0$. *Let* $(\bar{R}_h) \subset \mathrm{SO}(3)$ *be the sequence of rotations provided by Theorem 3.1.3–(i). Then, up to a subsequence,* $\bar{R}_h \to \bar{R} \in \mathcal{R}$.

Proof. Let $\bar{R}_h$, u_h, and v_h be the sequences given by Theorem 3.1.3–(i). Up to a subsequence, $\bar{R}_h \to \bar{R}$ for some $\bar{R} \in \mathrm{SO}(3)$. To prove that $\bar{R} \in \mathcal{R}$ pick a rotation $R \in \mathrm{SO}(3)$ and consider the test deformation

$$\hat{y}_h(x', x_3) := R \begin{pmatrix} x' \\ hx_3 \end{pmatrix}. \tag{4.5}$$

The elastic energy of $\hat{y}_h$ is zero, so

$$\begin{aligned} J_h(y_h) - J_h(\hat{y}_h) &\geq \int_\Omega f_h \cdot \hat{y}_h \, dx - \int_\Omega f_h \cdot y_h \, dx \\ &= -h^2 \int_S f \cdot \bar{R}_h \begin{pmatrix} \max\{\sqrt{D_h}, D_h/h^2\} u_h \\ h^{-1}\sqrt{D_h} v_h \end{pmatrix} dx' \\ &\quad + h^2 \int_S f \cdot (R - \bar{R}_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx'. \end{aligned}$$

Here, we have used that f does not depend on x_3 and the symmetry of $(-1/2, 1/2)$ to deduce that

$$\int_\Omega f \cdot R \begin{pmatrix} 0 \\ x_3 \end{pmatrix} dx = \int_\Omega f \cdot R_h \begin{pmatrix} 0 \\ x_3 \end{pmatrix} dx = 0.$$

Dividing by h^2 and passing to the limit we deduce by (4.4), Theorem 3.1.3–(i), and Remark 3.1.5

$$0 \geq \int_S f \cdot (R - \bar{R}) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx',$$

which gives $\bar{R} \in \mathcal{R}$. □

Proof of Theorem 4.2.1. The proof is divided in three steps, one for each possible elastic energy scaling. Let $\bar{R}_h$, u_h, and v_h be the sequences given by Theorem 3.1.3–(i). Up to subsequences, $\bar{R}_h \to \bar{R}$ in SO(3), with $\bar{R} \in \mathcal{R}$ by Lemma 4.2.2.

Step 1. We start by considering the case where $D_h h^{-4} \to 0$. Let $R \in \mathcal{R}$ and consider the test deformation

$$\hat{y}_h(x', x_3) := R \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + R \begin{pmatrix} -h^2 x_3 \nabla v^T \\ hv \end{pmatrix}. \tag{4.6}$$

By the same computation done in the proof of Theorem 3.1.3–(iv) we have that $E_h(\hat{y}_h) = O(h^4)$. Comparing the quasi-minimizing sequence with the test deformations and using that $R \in \mathcal{R}$ we get

$$\begin{aligned} J_h(y_h) - J_h(\hat{y}_h) &\geq O(h^4) + \int_\Omega f_h \cdot \hat{y}_h \, dx - \int_\Omega f_h \cdot y_h \, dx \\ &= O(h^4) - \int_S f_h \cdot \bar{R}_h \begin{pmatrix} \sqrt{D_h} u_h \\ h^{-1}\sqrt{D_h} v_h \end{pmatrix} dx' + \int_S f_h \cdot (R - \bar{R}_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ &\quad + \int_S f_h \cdot R \begin{pmatrix} 0 \\ hv \end{pmatrix} dx' \\ &\geq O(h^4) + h^3 \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} dx' - h\sqrt{D_h} \int_S f \cdot \bar{R}_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx'. \end{aligned} \tag{4.7}$$

Dividing by h^3 and passing to the limit, by (4.4) and the fact that $D_h h^{-4} \to 0$ we deduce that

$$0 \geq \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} dx' \qquad \forall R \in \mathcal{R}, \forall v \in C^\infty(\bar{S}),$$

and by density the same holds for every $v \in L^2(S)$. Since the map

$$v \in L^2(S) \mapsto \int_S R^T f \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dx'$$

is linear it must be identically zero, that is, $R^T f \cdot e_3 = 0$ for any $R \in \mathcal{R}$.

Step 2. We move now to the case where $D_h h^{-4} \to D > 0$. Let $R \in \mathcal{R}$ and $v \in C^\infty(\bar{S})$ and consider again the test deformation $\hat{y}_h$ as in (4.6). Arguing as in (4.7), we deduce that

$$J_h(y_h) - J_h(\hat{y}_h) \geq h^3 \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} dx' - h\sqrt{D_h} \int_S f \cdot \bar{R}_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O(h^4).$$

Dividing by h^3 and passing to the limit we deduce by (4.4) that

$$0 \geq \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} dx' - \sqrt{D} \int_S f \cdot \bar{R} \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' \qquad \forall R \in \mathcal{R}, \forall v \in C^\infty(\bar{S}),$$

and by density the same holds for every $v \in L^2(S)$. Arguing as before, we conclude by linearity.

Step 3. Finally, we discuss the case $D_h h^{-4} \to +\infty$. Let $v \in C^\infty(\bar{S}) \cap \mathcal{A}_{\det}$ (see Section 2.2.2 for the definition of $\mathcal{A}_{\det}$). By Theorem 2.2.10 there exists $\tilde{u}_h \in W^{2,\infty}(S;\mathbb{R}^2)$ such that

$$\tilde{y}_h(x') := \begin{pmatrix} x' \\ 0 \end{pmatrix} + \begin{pmatrix} h^{-2} D_h \tilde{u}_h \\ h^{-1}\sqrt{D_h} v \end{pmatrix}$$

is an isometric immersion, i.e., $\nabla' \tilde{y}_h^T \nabla' \tilde{y}_h = \mathrm{Id}$. Note that by (2.12) and the fact that $h^{-1}\sqrt{D_h} \to 0$, we have $\|\tilde{u}_h\|_{W^{2,\infty}} \leq C$. Let $R \in \mathcal{R}$ and define

$$\hat{y}_h := R\tilde{y}_h + hx_3 R\nu_h,$$

where $\nu_h := \partial_1 \tilde{y}_h \wedge \partial_2 \tilde{y}_h$. Arguing as in the proof of Theorem 3.1.3–(iii) we have $E_h(\hat{y}_h) = O(D_h)$. Comparing the test deformation $\hat{y}_h$ with the minimizing sequence, using that $R \in \mathcal{R}$, and that (2.12) holds true, we get

$$\begin{aligned} J_h(y_h) - J_h(\hat{y}_h) &\geq \int_\Omega f_h \cdot \hat{y}_h \, dx - \int_\Omega f_h \cdot y_h \, dx + O(D_h) \\ &= -\int_S f_h \cdot \bar{R}_h \begin{pmatrix} h^{-2} D_h u_h \\ h^{-1}\sqrt{D_h} v_h \end{pmatrix} dx' + \int_S f_h \cdot (R - \bar{R}_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ &\quad + \int_S f_h \cdot R \begin{pmatrix} h^{-2} D_h \tilde{u}_h \\ h^{-1}\sqrt{D_h} v \end{pmatrix} dx' + O(D_h) \\ &\geq h\sqrt{D_h} \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} dx' - h\sqrt{D_h} \int_S f \cdot \bar{R}_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O(D_h). \end{aligned}$$

Dividing by $h\sqrt{D_h}$ and passing to the limit we obtain that for every $R \in \mathcal{R}$ and for every $v \in C^\infty(\bar{S}) \cap \mathcal{A}_{\det}$ we have

$$0 \geq \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} dx' - \int_S f \cdot \bar{R} \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx'. \tag{4.8}$$

Since S satisfies condition (2.17), Corollary 2.2.13 ensures that (4.8) actually holds for any $v \in \mathcal{A}_{\det}$. By Lemma 4.2.2, $\bar{R} \in \mathcal{R}$, hence, choosing $R := \bar{R}$ we have once again that $\bar{v}$ maximizes the linear map

$$v \mapsto \int_S \bar{R}^T f \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dx'$$

on $\mathcal{A}_{\det}$. Note that $\mathcal{A}_{\det}$ is not a linear space. However, if $v \in \mathcal{A}_{\det}$, then $\lambda v \in \mathcal{A}_{\det}$ for any $\lambda \in \mathbb{R}$. Therefore, we conclude that

$$\int_S \bar{R}^T f \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} \, dx' = 0 \qquad \forall\, v \in \mathcal{A}_{\det}.$$

Going back to (4.8), we deduce that

$$0 \geq \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} \, dx' \qquad \forall\, R \in \mathcal{R}, \forall\, v \in \mathcal{A}_{\det}.$$

Hence, by linearity the same holds for every $v \in \overline{\operatorname{span} \mathcal{A}_{\det}}$, where the closure is taken in $L^2(S)$. The conclusion follows from Theorem 2.2.14. □

The rest of the section is devoted to the proof of Theorem 4.1.2–4.1.4.

Proof of Theorem 4.1.2. Firstly, observe that $J_h(y_h) \leq Ch^2$. Indeed, using the test deformation (4.5) we have

$$\inf_y J_h(y) \leq J_h(\hat{y}_h) = h^2 \int_S f \cdot R \begin{pmatrix} x' \\ 0 \end{pmatrix} \, dx' \leq Ch^2.$$

By Proposition 3.2.6 there is a constant rotation $Q_h \in \mathrm{SO}(3)$ such that

$$\|\nabla_h y_h - Q_h\|_{L^2}^2 \leq Ch^{-2} E_h(y_h).$$

We now define

$$\tilde{y}_h := Q_h^T y_h - \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + c_h$$

where c_h is chosen so that $\tilde{y}_h$ has zero average. By Poincaré–Wirtinger inequality, one obtains a bound from above on the elastic energy as follows

$$\begin{aligned} E_h(y_h) = J_h(y_h) + \int_\Omega f_h \cdot y_h \, dx &\leq Ch^2 + h^2 \int_\Omega f \cdot Q_h \tilde{y}_h \, dx + h^2 \int_S f \cdot \begin{pmatrix} x' \\ 0 \end{pmatrix} \, dx' \\ &\leq Ch^2 + Ch^2 \|\nabla_h y_h - Q_h\|_{L^2} \leq Ch^2 + Ch(E_h(y_h))^{\frac{1}{2}}. \end{aligned}$$

Thus, by a simple application of Young's inequality, we get $E_h(y_h) \leq Ch^2$. Assume now that $R^T f \cdot e_3 \neq 0$ for some $R \in \mathcal{R}$. It follows that

$$\liminf_{h \to 0} \frac{1}{h^2} E_h(y_h) = e > 0,$$

otherwise, defining $D_h := E_h(y_h)$, by Theorem 4.2.1 we would conclude that $R^T f \cdot e_3 = 0$ for every optimal rotation $R \in \mathcal{R}$, contradicting the assumption. By Theorem 3.1.1–(i), there exists $\bar{y} \in W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$ such that, up to a subsequence, $y_h \to \bar{y}$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $\nabla_h y_h \to \begin{pmatrix} \nabla' \bar{y} & \nu \end{pmatrix}$, where $\nu := \partial_1 \bar{y} \wedge \partial_2 \bar{y}$. By a standard Γ-convergence argument, being y_h quasi-minimizing, we deduce that

$$\frac{1}{h^2} J_h(y_h) \to J^{\mathrm{K}}(\bar{y}) = \inf J^{\mathrm{K}}.$$

In particular, since the loading term is continuous, we deduce by the Γ-convergence of $h^{-2}E_h$ that

$$\frac{1}{h^2}E_h(y_h) \to E^{\mathrm{K}}(\bar{y}) = e > 0.$$

This implies that $\bar{y} \neq R\begin{pmatrix} x' \\ 0 \end{pmatrix}$ for every $R \in \mathrm{SO}(3)$, so condition (S1) is not satisfied. □

We move now to the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. The proof follows the steps of [LM09, Theorem 4]. Arguing as in the proof of Theorem 4.1.2 we get $E_h(y_h) \leq Ch^2$.

Step 1. Firstly, suppose by contradiction that $h^{-2}E_h(y_h) \to e > 0$. In this case we can argue as in the proof of Theorem 4.1.2 to deduce that $y_h \to \bar{y}$ in $W^{1,2}(S;\mathbb{R}^3)$, $E^{\mathrm{K}}(\bar{y}) = e > 0$ and $J^{\mathrm{K}}(\bar{y}) = 0$ contradicting the stability condition (S1).

Step 2. Suppose now that $h^{-2}E_h(y_h) \to 0$ and $h^{-4}E_h(y_h) \to +\infty$. We show that this gives a contradiction. Set $D_h := E_h(y_h)$. Let $\bar{R}_h$, u_h, and v_h be the sequences given by Theorem 3.1.3–(i). By Lemma 4.2.2, up to a subsequence $R_h \to \bar{R} \in \mathcal{R}$ thus, at least for $h \ll 1$, the projection $P(R_h)$ of R_h onto $\mathcal{R}$ is well-defined. Define $d_h := \mathrm{dist}_{\mathrm{SO}(3)}(R_h, \mathcal{R})$ (see (2.5) for the definition of $\mathrm{dist}_{\mathrm{SO}(3)}$). Let $W_h \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$ such that $P(R_h)W_h \in \mathrm{N}\mathcal{R}_{P(R_h)}$, $|W_h| = 1$, and $R_h = P(R_h)e^{d_h W_h}$. Recall that $\mathrm{N}\mathcal{R}_{P(R_h)}$ is the normal space to $\mathcal{R}$ at the point $P(R_h)$ (see (2.3) for the definition of normal space). By Lemma 2.2.17, up to a subsequence, $W_h \to \bar{W}_1$ with $|\bar{W}_1| = 1$ and $\bar{R}\bar{W}_1 \in \mathrm{N}\mathcal{R}_{\bar{R}}$. Indeed, in the setting of this chapter, (F^2) is trivially true since $\mathcal{R}_h = \mathcal{R}$ for every h.

We now show that $d_h = O(h^{-1}\sqrt{D_h})$. Let $v \in \mathcal{A}_{\det} \cap C^\infty(\bar{S})$ and $\tilde{u}_h \in W^{2,\infty}(S)$ given by Theorem 2.2.10 so that the map

$$\tilde{y}_h(x') := \begin{pmatrix} x' \\ 0 \end{pmatrix} + \begin{pmatrix} h^{-2}D_h\tilde{u}_h \\ h^{-1}\sqrt{D_h}v \end{pmatrix}$$

is an isometric immersion. Note that, since $h^{-1}\sqrt{D_h} \to 0$, we have the uniform bound $\|\tilde{u}_h\|_{W^{2,\infty}} \leq C$. Consider the test deformation

$$\hat{y}_h := P(R_h)\tilde{y}_h + hx_3 P(R_h)\nu_h,$$

Reasoning as in the proof of Theorem 3.1.3–(iii), we have $E_h(\hat{y}_h) = O(D_h)$. Thus,

$$\begin{aligned} J_h(y_h) - J_h(\hat{y}_h) &\geq \int_\Omega f_h \cdot \hat{y}_h \, dx - \int_\Omega f_h \cdot y_h \, dx + O(D_h) \\ &= -\int_S f \cdot R_h \begin{pmatrix} D_h u_h \\ h\sqrt{D_h}v_h \end{pmatrix} dx' + h^2 \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ &\quad + \int_S f \cdot P(R_h) \begin{pmatrix} D_h \tilde{u}_h \\ h\sqrt{D_h}v \end{pmatrix} dx' + O(D_h). \end{aligned} \tag{4.9}$$

As showed in (2.20), we have that

$$\int_S f \cdot P(R_h)W \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' = 0 \qquad \forall\, W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}. \tag{4.10}$$

Expanding the exponential map $e^{d_h W_h}$, recalling that by Theorem 4.2.1 we have $P(R_h)^T f \cdot e_3 = 0$, and using both (2.12) and (4.10), we get from (4.9)

$$\begin{aligned} J_h(y_h) - J_h(\hat{y}_h) \geq & -h^2 d_h^2 \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ & + h\sqrt{D_h} \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O(D_h, h^2 d_h^3) \\ \geq & -h^2 d_h^2 \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ & - h d_h \sqrt{D_h} \int_S f \cdot P(R_h) W_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O(D_h, h^2 d_h^3, h\sqrt{D_h} d_h^2). \end{aligned} \tag{4.11}$$

Suppose by contradiction that $hd_h/\sqrt{D_h} \to +\infty$. Then dividing (4.11) by $h^2 d_h^2$ we get

$$\begin{aligned} \frac{1}{h^2 d_h^2}(J_h(y_h) - J_h(\hat{y}_h)) \geq & - \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ & - \frac{\sqrt{D_h}}{hd_h} \int_S f \cdot P(R_h) W_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O\left(d_h, \frac{D_h}{h^2 d_h^2}, \frac{\sqrt{D_h}}{h}\right). \end{aligned} \tag{4.12}$$

Note that, by (4.4) we have

$$\limsup_{h\to 0} \frac{1}{h^2 d_h^2}(J_h(y_h) - J_h(\hat{y}_h)) = \limsup_{h\to 0} \frac{D_h}{h^2 d_h^2} \frac{h^4}{D_h} \frac{1}{h^4}(J_h(y_h) - J_h(\hat{y}_h)) \leq 0.$$

Passing to the limit in (4.12) we deduce that

$$0 \geq - \int_S f \cdot \bar{R} \bar{W}_1^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' > 0,$$

where the last inequality follows from the fact that $\bar{R}\bar{W}_1 \in \mathrm{N}\mathcal{R}_{\bar{R}}$ and $\bar{W}_1 \neq 0$ (see (2.3) and (2.20)). This gives a contradiction and proves that $d_h = O(h^{-1}\sqrt{D_h})$.

To conclude the proof of Step 2 we show now that condition (S2) is violated, getting a contradiction. Set

$$\bar{W} := \lim_{h\to 0} \frac{h}{\sqrt{D_h}} d_h W_h.$$

Since $\bar{R}\bar{W}_1 \in \mathrm{N}\mathcal{R}_{\bar{R}}$ we have $\bar{R}\bar{W} \in \mathrm{N}\mathcal{R}_{\bar{R}}$. We have that

$$\begin{aligned} & \frac{1}{D_h}(J_h(y_h) + h^2 F(P(R_h))) \\ & \quad = \frac{1}{D_h} E_h(y_h) - \frac{h^2}{D_h} \int_\Omega f \cdot y_h \, dx' + \frac{h^2}{D_h} \int_S f \cdot P(R_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ & \quad = \frac{1}{D_h} E_h(y_h) + \frac{h^2}{D_h} \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ & \qquad - \frac{h^2}{D_h} \int_S f \cdot R_h \begin{pmatrix} h^{-2} D_h u_h \\ h^{-1}\sqrt{D_h} v_h \end{pmatrix} dx'. \end{aligned}$$

Expanding twice the exponential map, recalling that $P(R_h)^T f \cdot e_3 = 0$, and by (4.10) we get

$$
\begin{aligned}
&\frac{1}{D_h}(J_h(y_h) + h^2 F(P(R_h))) \\
&\quad = \frac{1}{D_h} E_h(y_h) - \frac{h^2 d_h^2}{D_h} \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' - \int_S f \cdot R_h \begin{pmatrix} u_h \\ 0 \end{pmatrix} dx' \\
&\qquad + \frac{h}{\sqrt{D_h}} \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O\left(\frac{h^2 d_h^3}{D_h}\right) \\
&\quad = \frac{1}{D_h} E_h(y_h) - \frac{h^2 d_h^2}{D_h} \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' - \int_S f \cdot R_h \begin{pmatrix} u_h \\ 0 \end{pmatrix} dx' \\
&\qquad - \frac{h d_h}{\sqrt{D_h}} \int_S f \cdot P(R_h) W_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O\left(\frac{h^2 d_h^3}{D_h}, \frac{h d_h^2}{\sqrt{D_h}}\right).
\end{aligned}
$$

We denote by $\bar{v}$ and $\bar{u}$ the limits of v_h and u_h, respectively. Note that by Theorem 3.1.3–(i), $(\bar{u}, \bar{v}) \in \mathcal{B}^{\text{lin}}_{\text{iso}}$. Since by definition $D_h^{-1} E_h(y_h) \to 1$, passing to the limit we get by Theorem 3.1.3–(ii)

$$
\begin{aligned}
&\liminf_{h\to 0} \frac{1}{D_h}(J_h(y_h) + h^2 F(P(R_h))) \\
&\quad = 1 - \int_S f \cdot \bar{R} \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix} dx' - \int_S f \cdot \bar{R}\bar{W} \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' \\
&\qquad - \int_S f \cdot \bar{R}\bar{W}^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \geq J^{\text{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) \geq 0,
\end{aligned} \tag{4.13}
$$

where the last inequality follows from (S2).

Let $\hat{y}_h$ be the test deformation in (4.6) with $v \in C^\infty(\bar{S})$ and $R \in \mathcal{R}$. Reasoning as in the proof of Theorem 3.1.3–(iv), we have that $E_h(\hat{y}_h) = O(h^4)$, hence

$$
\frac{1}{D_h}(J_h(\hat{y}_h) + h^2 F(P(R_h))) \leq -\frac{h^3}{D_h} \int_S f \cdot R \begin{pmatrix} 0 \\ v \end{pmatrix} dx' + O\left(\frac{h^4}{D_h}\right) = O\left(\frac{h^4}{D_h}\right) \to 0,
$$

where we used that $F(P(R_h)) = F(R)$ for every $R \in \mathcal{R}$ and that $R^T f \cdot e_3 = 0$. In particular, by the quasi-minimizing property of y_h

$$
\limsup_{h\to 0} \frac{1}{D_h}(J_h(y_h) + h^2 F(P(R_h))) \leq \limsup_{h\to 0} \frac{1}{D_h}(J_h(y_h) - J_h(\hat{y}_h)) = 0.
$$

Hence, all the inequalities in (4.13) are in fact equalities, and we have $E^{\text{VK}}(\bar{u}, \bar{v}) = 1$ and $J^{\text{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) = 0$. Since $(\bar{u}, \bar{v}) \in \mathcal{B}^{\text{lin}}_{\text{iso}}$, this contradicts (S2).

Step 3. By the previous steps, we obtain that $E_h(y_h) \leq Ch^4$. Define d_h and W_h as in Step 2. We prove now that $d_h = O(h)$. The argument is similar to the one already seen. Consider the test deformation (4.6) with $v \in C^\infty(\bar{S})$ and $R := P(R_h)$. Arguing as in the proof of Theorem 3.1.3–(iv) we have $E_h(\hat{y}_h) = O(h^4)$, thus, expanding the exponential and recalling that $F(RW) = 0$ for every $R \in \mathcal{R}$ and

$W \in \mathbb{R}^{3\times 3}_{\text{skew}}$

$$
\begin{aligned}
J_h(y_h) - J_h(\hat{y}_h) &\geq \int_\Omega f_h \cdot \hat{y}_h \, dx - \int_\Omega f_h \cdot y_h \, dx + O(h^4) \\
&= -h^2 \int_S f \cdot R_h \begin{pmatrix} h^2 u_h \\ h v_h \end{pmatrix} dx' + h^2 \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\
&\quad + h^2 \int_S f \cdot P(R_h) \begin{pmatrix} 0 \\ h v \end{pmatrix} dx' + O(h^4) \\
&\geq -h^2 d_h^2 \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\
&\quad + h^3 \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O(h^4, h^2 d_h^3) \\
&\geq -h^2 d_h^2 \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\
&\quad - h^3 d_h \int_S f \cdot P(R_h) W_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O(h^4, h^2 d_h^3, h^3 d_h^2).
\end{aligned}
\tag{4.14}
$$

Suppose by contradiction that $d_h/h \to +\infty$. Then, dividing (4.14) by $h^2 d_h^2$ and passing to the limit we deduce that

$$
0 \geq \int_S f \cdot \bar{R} \bar{W}_1^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' > 0,
$$

where the last inequality follows from the fact that $0 \neq \bar{R}\bar{W}_1 \in \mathrm{N}\mathcal{R}_{\bar{R}}$. This provides the desired contradiction.

Define as before

$$
\bar{W} := \lim_{h\to 0} \frac{d_h}{h} W_h.
$$

Finally, expanding the exponential map

$$
\begin{aligned}
&\frac{1}{h^4}(J_h(y_h) + h^2 F(P(R_h))) = \\
&\quad = \frac{1}{h^4} E_h(y_h) - \frac{1}{h^2} \int_\Omega f \cdot y_h \, dx' + \frac{1}{h^2} \int_S f \cdot P(R_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\
&\quad = \frac{1}{h^4} E_h(y_h) + \frac{1}{h^2} \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' - \frac{1}{h^2} \int_S f \cdot R_h \begin{pmatrix} h^2 u_h \\ h v_h \end{pmatrix} dx' \\
&\quad = \frac{1}{h^4} E_h(y_h) - \frac{d_h^2}{h^2} \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' - \int_S f \cdot R_h \begin{pmatrix} u_h \\ 0 \end{pmatrix} dx' \\
&\qquad + \frac{1}{h} \int_S f \cdot (P(R_h) - R_h) \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O\left(\frac{d_h^3}{h^2}\right) \\
&\quad = \frac{1}{h^4} E_h(y_h) - \frac{d_h^2}{h^2} \int_S f \cdot P(R_h) W_h^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\
&\qquad - \int_S f \cdot R_h \begin{pmatrix} u_h \\ 0 \end{pmatrix} dx' - \frac{d_h}{h} \int_S f \cdot P(R_h) W_h \begin{pmatrix} 0 \\ v_h \end{pmatrix} dx' + O\left(\frac{d_h^3}{h^2}, \frac{d_h^2}{h}\right),
\end{aligned}
$$

so that

$$\liminf_{h\to 0} \frac{1}{h^4}(J_h(y_h) + h^2 F(P(R_h))) \geq J^{\mathrm{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}).$$

Let (u, v, R, W) be an admissible quadruplet. Construct a recovery sequence $(\tilde{y}_h)$ for u and v as in Theorem 3.1.3–(iv)–(c). The sequences of rescaled displacements for the recovery sequence, defined as in (3.3)–(3.4), are denoted by $\tilde{u}_h$ and $\tilde{v}_h$. We have

$$\begin{aligned} J^{\mathrm{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) &\leq \liminf_{h\to 0} \frac{1}{h^4}(J_h(y_h) + h^2 F(P(R_h))) \\ &\leq \limsup_{h\to 0} \frac{1}{h^4}(\inf_y J_h(y) + h^2 F(P(R_h))) \\ &\leq \limsup_{h\to 0} \frac{1}{h^4}(J_h(Re^{hW}\tilde{y}_h) + h^2 F(R)). \end{aligned}$$

To conclude it is sufficient to prove that

$$\limsup_{h\to 0} \frac{1}{h^4}(J_h(Re^{hW}\tilde{y}_h) + F(R)) = J^{\mathrm{VK}}(u, v, R, W).$$

Expanding the expression of J_h we have

$$\begin{aligned} \frac{1}{h^4}(J_h(Re^{hW}\tilde{y}_h) + h^2 F(R)) &= \frac{1}{h^4} E_h(Re^{hW}\tilde{y}_h) - \frac{1}{h^2}\int_\Omega f \cdot Re^{hW}\tilde{y}_h \, dx \\ &\quad + \frac{1}{h^2}\int_S f \cdot R \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ &= \frac{1}{h^4} E_h(\tilde{y}_h) - \frac{1}{h^2}\int_S f \cdot Re^{hW} \begin{pmatrix} h^2 \tilde{u}_h \\ h\tilde{v}_h \end{pmatrix} dx' + \frac{1}{h^2}\int_S f \cdot (R - Re^{hW}) \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ &= \frac{1}{h^4} E_h(\tilde{y}_h) - \int_S f \cdot Re^{hW} \begin{pmatrix} \tilde{u}_h \\ 0 \end{pmatrix} dx' - \int_S f \cdot RW \begin{pmatrix} 0 \\ \tilde{v}_h \end{pmatrix} dx' \\ &\quad - \int_S f \cdot RW^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' + O(h) \to J^{\mathrm{VK}}(u, v, R, W), \end{aligned}$$

ending the proof of the minimality. $\square$

We conclude the section by proving Theorem 4.1.4.

Proof of Theorem 4.1.4. Suppose by contradiction that there exists an admissible quadruplet $(\bar{u}, \bar{v}, \bar{R}, \bar{W})$ such that $(\bar{u}, \bar{v}) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ and $J^{\mathrm{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) < 0$. Let $\delta > 0$ and $\tilde{v} \in C^\infty(\bar{S})$ such that $\|\bar{v} - \tilde{v}\|_{W^{2,2}} \leq \delta$. Let $1 \gg \varepsilon > 0$. By Theorem 2.2.9, there is $u_\varepsilon \in W^{2,2}(S; \mathbb{R}^2)$ such that

$$y_\varepsilon(x') := \begin{pmatrix} x' + \varepsilon^2 u_\varepsilon \\ \varepsilon\tilde{v} \end{pmatrix}$$

is an isometric immersion and

$$\|u_\varepsilon\|_{W^{2,2}} \leq C\left(\|\nabla'\tilde{v}\|_{L^\infty}\|(\nabla')^2\tilde{v}\|_{L^2} + \|\nabla'\tilde{v}\|^2_{L^2}\right).$$

It follows that along a non-relabelled subsequence $u_\varepsilon \rightharpoonup u$ in $W^{2,2}(S;\mathbb{R}^2)$ for some $u \in W^{2,2}(S;\mathbb{R}^2)$. Moreover, since $\nabla' y_\varepsilon^T \nabla' y_\varepsilon = \mathrm{Id}$, we have

$$0 = \varepsilon^2 \left(\nabla' u_\varepsilon^T + \nabla' u_\varepsilon + \nabla' \tilde{v} \otimes \nabla' \tilde{v} \right) + o(\varepsilon^2),$$

where $o(\varepsilon^2)$ has to be intended in the L^2 sense. Dividing by ε^2 and passing to the limit we deduce that $(u, \tilde{v}) \in \mathcal{B}_{\mathrm{iso}}^{\mathrm{lin}}$. Moreover,

$$\begin{aligned} \mathrm{sym}(\nabla' u - \nabla' \bar{u}) &= 2(\nabla' \tilde{v} \otimes \nabla' \tilde{v} - \nabla' \bar{v} \otimes \nabla' \bar{v}) \\ &= 2(\nabla'(\tilde{v} - \bar{v}) \otimes \nabla' \tilde{v} - \nabla' \bar{v} \otimes \nabla'(\bar{v} - \tilde{v})). \end{aligned}$$

Hence, by Korn's inequality, there exists $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $\eta \in \mathbb{R}^2$ such that

$$\|u - \bar{u} - Ax' - \eta\|_{L^2} \le C\delta. \tag{4.15}$$

Consider the deformation

$$\bar{y}_\varepsilon(x') := \bar{R} e^{\varepsilon \bar{W}} y_\varepsilon \in \mathcal{A}_{\mathrm{iso}}.$$

We have

$$\nabla' \bar{y}_\varepsilon = \bar{R} e^{\varepsilon \bar{W}} \left(\begin{pmatrix} e_1 & e_2 \end{pmatrix} + \begin{pmatrix} \varepsilon^2 \nabla' u_\varepsilon \\ \varepsilon \nabla' \tilde{v} \end{pmatrix} \right)$$

and

$$\nu_\varepsilon = \partial_1 \bar{y}_\varepsilon \wedge \partial_2 \bar{y}_\varepsilon = \bar{R} e^{\varepsilon \bar{W}} \left(e_3 - \varepsilon \begin{pmatrix} \nabla' \tilde{v}^T \\ 0 \end{pmatrix} \right) + O(\varepsilon^2).$$

It follows that

$$\nabla' \nu_\varepsilon = -\varepsilon \bar{R} e^{\varepsilon \bar{W}} \begin{pmatrix} (\nabla')^2 \tilde{v} \\ 0 \end{pmatrix} + O(\varepsilon^2)$$

and

$$(\nabla' \bar{y}_\varepsilon)^T \nabla' \nu_\varepsilon = -\varepsilon (\nabla')^2 \tilde{v} + O(\varepsilon^2).$$

Thus, by condition (S1),

$$\begin{aligned} 0 \le J^{\mathrm{K}}(\bar{y}_\varepsilon) &= \int_S \bar{Q}((\nabla' \bar{y}_\varepsilon)^T \nabla' \nu_\varepsilon)\, dx' - \int_S f \cdot \bar{y}_\varepsilon \, dx' \\ &= \varepsilon^2 \int_S \bar{Q}((\nabla')^2 \tilde{v})\, dx' - \int_S f \cdot \bar{R} e^{\varepsilon \bar{W}} \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' - \varepsilon \int_S f \cdot \bar{R} e^{\varepsilon \bar{W}} \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix} dx' \\ &\quad - \varepsilon^2 \int_S f \cdot \bar{R} e^{\varepsilon \bar{W}} \begin{pmatrix} u_\varepsilon \\ 0 \end{pmatrix} dx' + o(\varepsilon^2). \end{aligned}$$

By Theorem 4.1.2 we have $\bar{R}^T f \cdot e_3 = 0$. Expanding the exponential around the identity and recalling that $F(\bar{R}W) = 0$ for every $W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$, we get

$$\begin{aligned} 0 \le J^{\mathrm{K}}(\bar{y}_\varepsilon) \le \varepsilon^2 \int_S \bar{Q}((\nabla')^2 \tilde{v})\, dx' - F(\bar{R}) - \varepsilon^2 \int_S f \cdot \bar{R} (\bar{W})^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' \\ - \varepsilon^2 \int_S f \cdot \bar{R} \bar{W} \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix} dx' - \varepsilon^2 \int_S f \cdot \bar{R} \begin{pmatrix} u_\varepsilon \\ 0 \end{pmatrix} dx' + o(\varepsilon^2). \end{aligned}$$

Dividing by ε^2 and using the fact that $F(\bar{R}) \geq 0$ by Lemma 2.2.24, passing to the limit we deduce that $0 \leq J^{\mathrm{VK}}(u, \tilde{v}, \bar{R}, \bar{W})$. Hence, by definition of $\tilde{v}$ and (4.15) we get

$$0 \leq J^{\mathrm{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) + \int_S f \cdot \bar{R} \begin{pmatrix} Ax' + \eta \\ 0 \end{pmatrix} dx' + C\delta = J^{\mathrm{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) + C\delta,$$

where in the last equality we have used (4.2) and the fact that $F(\bar{R}W) = 0$ for every $W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$. Since δ is arbitrary we reach a contradiction.

We now prove that (S2) holds for $J^{\mathrm{VK}}_\varepsilon$. Suppose that there is an admissible quadruplet $(\bar{u}, \bar{v}, \bar{R}, \bar{W})$ such that $(\bar{u}, \bar{v}) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ and $J^{\mathrm{VK}}_\varepsilon(\bar{u}, \bar{v}, \bar{R}, \bar{W}) \leq 0$ for some $\varepsilon > 0$. We show that $\bar{v}$ is affine. Let

$$K := \int_S f \cdot \bar{R} \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix} dx' + \int_S f \cdot \bar{R}\bar{W} \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' + \int_S \bar{R}(\bar{W})^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx'.$$

If $K \leq 0$, since

$$0 \geq J^{\mathrm{VK}}_\varepsilon(\bar{u}, \bar{v}, \bar{R}, \bar{W}) = E^{\mathrm{VK}}(\bar{u}, \bar{v}) - (1-\varepsilon)K \geq E^{\mathrm{VK}}(\bar{u}, \bar{v}),$$

we get that $E^{\mathrm{VK}}(\bar{u}, \bar{v}) = 0$, thus, $\bar{v}$ is affine. Conversely, if $K > 0$ we deduce that

$$J^{\mathrm{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) = J^{\mathrm{VK}}_\varepsilon(\bar{u}, \bar{v}, \bar{R}, \bar{W}) - \varepsilon K < 0,$$

which gives a contradiction. $\square$

4.3 Attainment of the infimum of J^{VK}

In this last section, we prove Theorem 4.1.5. The stability condition (S2) assures that all configurations in $\mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ with zero total energy have zero Von Kármán's elastic energy, i.e., v is affine. However, we do not expect that all affine functions have zero total energy, unless $f = 0$. In the following series of results, we study the specific structure of such affine minimizers. We recall that we assume f not to be identically zero. Given an optimal rotation $R \in \mathcal{R}$, in the following results we often use the coefficients $a(R)$, $b(R)$, and $c(R)$ as defined in (2.21)–(2.23).

Proposition 4.3.1. *Suppose that (S2) and (C) hold, and $\dim \mathcal{R} = 1$. Let (u, v, R, W) be an admissible quadruplet such that $(u, v) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ and $J^{\mathrm{VK}}(u, v, R, W) = 0$. Then $W = 0$ and there are $\lambda, \delta \in \mathbb{R}$, $\eta \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ such that, if $a(R) > 0$, then*

$$\begin{aligned} v(x') &= -\lambda \frac{c(R)}{a(R)} x_1 + \lambda x_2 + \delta, \\ u(x') &= -\frac{\lambda^2}{2} \begin{pmatrix} \dfrac{b(R)}{a(R)} x_1 - \dfrac{c(R)}{a(R)} x_2 \\ -\dfrac{c(R)}{a(R)} x_1 + x_2 \end{pmatrix} + Ax' + \eta, \end{aligned}$$

whereas, if $b(R) > 0$, then

$$v(x') = \lambda x_1 - \lambda \frac{c(R)}{b(R)} x_2 + \delta,$$

$$u(x') = -\frac{\lambda^2}{2} \begin{pmatrix} x_1 - \dfrac{c(R)}{b(R)} x_2 \\ -\dfrac{c(R)}{b(R)} x_1 + \dfrac{a(R)}{b(R)} x_2 \end{pmatrix} + Ax' + \eta.$$

Proof. The stability condition (S2) implies that $v = \lambda_1 x_1 + \lambda_2 x_2 + \delta$. Since $(u, v) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$, we deduce that

$$u(x') = -\frac{1}{2} \begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 \\ \lambda_1\lambda_2 & \lambda_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + Ax' + \eta,$$

for some $\eta \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$. Now for any $\bar{A} \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $\bar{\eta} \in \mathbb{R}^2$, and $\bar{\delta} \in \mathbb{R}$ we have

$$J^{\mathrm{VK}}(u + \bar{A}x' + \bar{\eta}, v + \bar{\delta}, R, W) = J^{\mathrm{VK}}(u, v, R, W).$$

This follows from assumption (4.2) and the fact that $F(RW) = 0$ for any $W \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}$. In particular, we can suppose A, δ, and η to be zero.

Suppose $a(R) \neq 0$ (the proof for the case $b(R) \neq 0$ is analogous). We write a, b, c in place of $a(R), b(R), c(R)$ in order to streamline the exposition. By Corollary 2.2.27 in this case W is of the form

$$W = \begin{pmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & \dfrac{c}{a} W_{13} \\ -W_{13} & -\dfrac{c}{a} W_{13} & 0 \end{pmatrix}.$$

Let us define $P(W) := F(RW^2)$ and $J_{\min} := J^{\mathrm{VK}}(u, v, R, W)$. With some simple expansion (recall that $ab - c^2 = 0$ by Proposition 2.2.26, since $f \neq 0$) we have

$$\begin{aligned} J_{\min} &= \frac{1}{2} \int_S f \cdot R \begin{pmatrix} \lambda_1^2 x_1 + \lambda_1\lambda_2 x_2 \\ \lambda_1\lambda_2 x_1 + \lambda_2^2 x_2 \\ 0 \end{pmatrix} dx' \\ &\quad - \int_S f \cdot R \begin{pmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & \frac{c}{a} W_{13} \\ -W_{13} & -\frac{c}{a} W_{13} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_1 x_1 + \lambda_2 x_2 \end{pmatrix} dx' - P(W) \\ &= \frac{1}{2}(\lambda_1^2 a + 2\lambda_1\lambda_2 c + \lambda_2^2 b) - \lambda_1 W_{13}(a + b) - \lambda_2 W_{13} c \left(1 + \frac{b}{a}\right) - P(W). \end{aligned}$$

If we define

$$M := \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad B := W_{13} \begin{pmatrix} a + b \\ c\left(1 + \frac{b}{a}\right) \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$

then we have

$$J_{\min} = \frac{1}{2} \Lambda^T M \Lambda - B\Lambda - P(W).$$

By Lemma 2.2.25 and Proposition 2.2.26 M is positive semidefinite and by (S2) Λ is a minimizer of the map

$$z \mapsto \frac{1}{2} z^T M z - Bz - P(W).$$

Thus, $M\Lambda = B$. Solving this system one easily gets that

$$\lambda_1 = -\frac{c}{a}\lambda_2 + W_{13}\left(1 + \frac{b}{a}\right).$$

To conclude we just need to prove that $W = 0$. Observe that

$$(W^2)' = -\begin{pmatrix} W_{12}^2 + W_{13}^2 & \frac{c}{a} W_{13}^2 \\ \frac{c}{a} W_{13}^2 & W_{12}^2 + \frac{b}{a} W_{13}^2 \end{pmatrix}.$$

Thus, by definition of $P(W)$,

$$P(W) = -(W_{12}^2 + W_{13}^2)a - 2bW_{13}^2 - bW_{12}^2 - \frac{b^2}{a} W_{13}^2.$$

Substituting the expression of λ_1 and $P(W)$ in $J_{\min}$ we get

$$J_{\min} = W_{12}^2(a+b) + W_{13}^2 \frac{1}{2a}(a+b)^2,$$

so that, since $a + b > 0$ and $J_{\min} = 0$, we deduce $W = 0$. □

To simplify the exposition, given f such that $R^T f \cdot e_3 = 0$ for every $R \in \mathcal{R}$ let us define

$$\mathcal{V}_R := \begin{cases} \left\{ v \in W^{2,2}(S) \colon v(x') = -\lambda \frac{c(R)}{a(R)} x_1 + \lambda x_2,\ \lambda \in \mathbb{R} \right\} & \text{if } a(R) \neq 0, \\ \left\{ v \in W^{2,2}(S) \colon v(x') = \lambda x_1 - \lambda \frac{c(R)}{b(R)} x_2,\ \lambda \in \mathbb{R} \right\} & \text{if } b(R) \neq 0, \end{cases}$$

$$\mathcal{U}_R := \left\{ u \in W^{1,2}(S;\mathbb{R}^2) \colon u(x') = -\frac{1}{2}(\nabla' v \otimes \nabla' v)x',\ v \in \mathcal{V}_R \right\}.$$

Lemma 4.3.2. *Suppose $R^T f \cdot e_3 = 0$ for every $R \in \mathcal{R}$ and $\dim \mathcal{R} = 1$. Let $R \in \mathcal{R}$. Then*

$$\int_S f \cdot R \begin{pmatrix} u \\ 0 \end{pmatrix} dx' = 0,$$

for every $u \in \mathcal{U}_R$.

Proof. Let $u \in \mathcal{U}_R$ and $v \in \mathcal{V}_R$ be such that $u(x') = -(\nabla' v \otimes \nabla' v)x'/2$. By (2.2) it is sufficient to prove that $\nabla' v \otimes \nabla' v = -(W^2)'$ for some $W \in T\mathcal{R}_R$.

Suppose $a(R) \neq 0$. Then

$$v(x') = -\lambda \frac{c(R)}{a(R)} x_1 + \lambda x_2$$

for some $\lambda \in \mathbb{R}$, so

$$\nabla' v \otimes \nabla' v = \lambda^2 \begin{pmatrix} \frac{b(R)}{a(R)} & -\frac{c(R)}{a(R)} \\ -\frac{c(R)}{a(R)} & 1 \end{pmatrix},$$

where we used Proposition 2.2.26. Then, defining

$$W := \lambda \begin{pmatrix} 0 & 0 & \frac{c(R)}{a(R)} \\ 0 & 0 & -1 \\ -\frac{c(R)}{a(R)} & 1 & 0 \end{pmatrix},$$

we easily get $\nabla' v \otimes \nabla' v = -(W^2)'$ and $W \in T\mathcal{R}_R$ by Proposition 2.2.26. The case $b(R) \neq 0$ can be treated similarly. □

Lemma 4.3.3. *Suppose that $R^T f \cdot e_3 = 0$ for every $R \in \mathcal{R}$ and $\dim \mathcal{R} = 1$. Let $R \in \mathcal{R}$ and $v \in \mathcal{V}_R$. Then for any $\bar{v} \in W^{2,2}(S)$ there is $\xi \in W^{1,2}(S;\mathbb{R}^2)$ such that*

$$\nabla' \xi^T + \nabla' \xi + \nabla' \bar{v} \otimes \nabla' v + \nabla' v \otimes \nabla' \bar{v} = 0$$

and

$$\int_S f \cdot R \begin{pmatrix} \xi \\ 0 \end{pmatrix} dx' = 0.$$

Proof. Suppose $a(R) \neq 0$ and let $\lambda \in \mathbb{R}$ be such that

$$v(x') = -\lambda \frac{c(R)}{a(R)} x_1 + \lambda x_2.$$

For $\bar{v} \in W^{2,2}(S)$ it is sufficient to define

$$\xi(x') := \lambda \bar{v}(x') \begin{pmatrix} -\frac{c(R)}{a(R)} \\ 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} \xi \\ 0 \end{pmatrix} = \lambda W \begin{pmatrix} 0 \\ 0 \\ \bar{v} \end{pmatrix} \quad \text{with} \quad W := \begin{pmatrix} 0 & 0 & -\frac{c(R)}{a(R)} \\ 0 & 0 & 1 \\ \frac{c(R)}{a(R)} & -1 & 0 \end{pmatrix} \in T\mathcal{R}_R$$

by Proposition 2.2.26. In particular,

$$\int_S f \cdot R \begin{pmatrix} \xi \\ 0 \end{pmatrix} dx' = \lambda \int_S f \cdot RW \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx'. \tag{4.16}$$

Define the map $\Phi(t) = Re^{tW}$ for $t \in \mathbb{R}$. By (2.4), $\Phi(t) \in \mathcal{R}$ for any $t \in \mathbb{R}$, therefore

$$\int_S f \cdot \Phi(t) \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' = 0 \qquad \forall\, t \in \mathbb{R},$$

since $\Phi(t)^T f \cdot e_3 = 0$. Differentiating with respect to t at $t = 0$, we deduce

$$\int_S f \cdot RW \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' = 0,$$

which gives the thesis by (4.16). □

Having all the previous results at our disposal we can show that J^{VK} enjoys some invariance properties.

Proposition 4.3.4. *Suppose $R^T f \cdot e_3 = 0$ for every $R \in \mathcal{R}$ and $\dim \mathcal{R} = 1$. Let $\bar{v} \in \mathcal{V}_R$ and $\bar{u} \in \mathcal{U}_R$ be such that*

$$\nabla' \bar{u}^T + \nabla' \bar{u} + \nabla' \bar{v} \otimes \nabla' \bar{v} = 0.$$

Then $J^{\mathrm{VK}}(u + \bar{u} + \xi, v + \bar{v}, R, W) = J^{\mathrm{VK}}(u, v, R, W)$ for every admissible quadruplet (u, v, R, W), where ξ is defined as in Lemma 4.3.3.

Proof. Since v is affine we immediately have that $(\nabla')^2(v + \bar{v}) = (\nabla')^2 v$. Moreover, by definition of ξ

$$\begin{aligned}&(\nabla'(u + \bar{u} + \xi))^T + \nabla'(u + \bar{u} + \xi) + \nabla'(v + \bar{v}) \otimes \nabla'(v + \bar{v}) \\ &\quad = (\nabla' u)^T + \nabla' u + \nabla' v \otimes \nabla' v.\end{aligned}$$

By Lemmas 4.3.2 and 4.3.3, to conclude we just need to show that

$$\int_S f \cdot RW \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' = 0.$$

This easily follows from the specific structure of $\bar{v}$. Indeed, suppose $a(R) \neq 0$ and let $\lambda \in \mathbb{R}$ be such that

$$\bar{v}(x') = -\lambda \frac{c(R)}{a(R)} x_1 + \lambda x_2.$$

Then

$$\begin{aligned}\int_S f \cdot RW \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' &= \lambda \left(-W_{13} c(R) + W_{13} c(R) - W_{23} \frac{c^2(R)}{a(R)} + W_{23} b(R) \right) \\ &= \lambda \left(-W_{23} \frac{c^2(R)}{a(R)} + W_{23} b(R) \right) = 0,\end{aligned}$$

since $a(R) b(R) = c^2(R)$ by Proposition 2.2.26. □

We are finally ready to give the proof of Theorem 4.1.5.

Proof of Theorem 4.1.5. Let (u_n, v_n, R_n, W_n) be a minimizing sequence for J^{VK}. Let $P_n^{\mathcal{V}}$ be the projection of $W^{2,2}(S)$ onto $\mathcal{V}_{R_n}$. By Proposition 4.3.4 and the fact that for every $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $\eta \in \mathbb{R}^2$, and $\delta \in \mathbb{R}$

$$J^{\mathrm{VK}}(u_n + Ax' + \eta, v_n + \delta, R_n, W_n) = J^{\mathrm{VK}}(u_n, v_n, R_n, W_n)$$

we can suppose that for all $n \in \mathbb{N}$

(i) $\displaystyle\int_S u_n \, dx' = 0$,

(ii) $\displaystyle\int_S v_n \, dx' = 0$,

(iii) $P_n^{\mathcal{V}}(v_n) = 0$,

(iv) $\int_S \operatorname{skew}(\nabla' u_n)\,dx' = 0.$

Up to a subsequence, we can always assume that $R_n \to R \in \mathcal{R}$.

Assume first that

$$\|u_n\|_{W^{1,2}} + \|v_n\|^2_{W^{2,2}} + |W_n|^2 \le C.$$

Then, up to a subsequence we have $u_n \rightharpoonup u$ in $W^{1,2}(S;\mathbb{R}^2)$, $v_n \rightharpoonup v$ in $W^{2,2}(S)$ and $W_n \to W$ with $RW \in \mathrm{N}\mathcal{R}_R$. By lower semicontinuity of J^{VK} we deduce that (u, v, R, W) is a minimizer of J^{VK}.

Suppose now by contradiction that

$$\|u_n\|_{W^{1,2}} + \|v_n\|^2_{W^{2,2}} + |W_n|^2 = \gamma_n^2 \to +\infty$$

and define $\bar{u}_n := \gamma_n^{-2} u_n$, $\bar{v}_n := \gamma_n^{-1} v_n$ and $\bar{W}_n := \gamma_n^{-1} W_n$. Then, up to a subsequence, we have $\bar{u}_n \rightharpoonup \bar{u}$ in $W^{1,2}(S;\mathbb{R}^2)$, $\bar{v}_n \rightharpoonup \bar{v}$ in $W^{2,2}(S)$ and $\bar{W}_n \to \bar{W}$ with $R\bar{W} \in \mathrm{N}\mathcal{R}_R$. Since $J^{\mathrm{VK}}(u_n, v_n, R_n, W_n) \le C$, we have

$$\begin{aligned} C \ge \gamma_n^4 \int_S \bar{Q}((\nabla'\bar{u}_n)^T + \nabla'\bar{u}_n + \nabla'\bar{v}_n \otimes \nabla'\bar{v}_n)\,dx' + \gamma_n^2 \int_S \bar{Q}((\nabla')^2\bar{v}_n)\,dx' \\ - \gamma_n^2 \int_S f \cdot R_n \begin{pmatrix} \bar{u}_n \\ 0 \end{pmatrix} dx' - \gamma_n^2 \int_S f \cdot R_n \bar{W}_n \begin{pmatrix} 0 \\ \bar{v}_n \end{pmatrix} dx' \\ - \gamma_n^2 \int_S f \cdot R_n (\bar{W}_n)^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx'. \end{aligned} \tag{4.17}$$

Dividing by γ_n^4 we get by the coercivity of $\bar{Q}$

$$\|(\nabla'\bar{u}_n)^T + \nabla'\bar{u}_n + \nabla'\bar{v}_n \otimes \nabla'\bar{v}_n\|_{L^2} \le \frac{C}{\gamma_n}. \tag{4.18}$$

Passing to the limit we deduce that $(\bar{u}, \bar{v}) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$. Moreover, dividing (4.17) by γ_n^2 and passing to the limit we get by lower semicontinuity that $0 \ge J^{\mathrm{VK}}(\bar{u}, \bar{v}, R, \bar{W})$. The stability condition (S2) implies that $J^{\mathrm{VK}}(\bar{u}, \bar{v}, R, \bar{W})$ is zero and $\bar{v}$ is affine. By Proposition 4.3.1 and the properties (i)–(iv) we deduce that $\bar{u} = 0$, $\bar{v} = 0$ and $\bar{W} = 0$. If we prove that $\bar{u}_n$ and $\bar{v}_n$ are strongly converging, then the proof is concluded since we would have

$$\|\bar{u}\|_{W^{1,2}} + \|\bar{v}\|^2_{W^{2,2}} + |\bar{W}|^2 = 1.$$

Dividing (4.17) by γ_n^2 and passing to the limit we have

$$0 \ge \limsup_{n\to\infty} \int_S \bar{Q}((\nabla')^2\bar{v}_n)\,dx'.$$

In particular, by the coercivity of Q (see Lemma 2.2.6) we get $(\nabla')^2\bar{v}_n \to 0$ in $L^2(S;\mathbb{R}^{2\times 2})$, giving the strong convergence of $\bar{v}_n$ in $W^{2,2}(S)$. By (4.18) we have that $\operatorname{sym}(\nabla'\bar{u}_n) \to 0$ in $L^2(S;\mathbb{R}^{2\times 2})$. By (iv) we can apply Korn's inequality to deduce that $\bar{u}_n \to 0$ strongly in $W^{1,2}(S;\mathbb{R}^2)$, concluding the proof of the first part.

Suppose now that (S2) fails. Let $(\bar{v}, \bar{u}) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ such that for some $\bar{R} \in \mathcal{R}$ and $\bar{R}\bar{W} \in \mathrm{N}\mathcal{R}_{\bar{R}}$ either $J^{\mathrm{VK}}(\bar{y}, \bar{v}, \bar{R}, \bar{W}) < 0$ or $J^{\mathrm{VK}}(\bar{y}, \bar{v}, \bar{R}, \bar{W}) = 0$ and $\bar{v}$ is not affine. In any of these two cases, we have

$$-\int_S f \cdot \bar{R} \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix} dx' - \int_S f \cdot \bar{R}\bar{W} \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' - \int_S f \cdot \bar{R}\bar{W}^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx' < 0.$$

In particular we have that $J^{\mathrm{VK}}_\varepsilon(\bar{u}, \bar{v}, \bar{R}, \bar{W}) < 0$ for every choice of $\varepsilon > 0$. Since for every $\gamma > 0$ we have that $(\gamma^2 \bar{u}, \gamma \bar{v}) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ and

$$\begin{aligned} J^{\mathrm{VK}}_\varepsilon(\gamma^2 \bar{u}, \gamma \bar{v}, \bar{R}, \gamma \bar{W}) &= \gamma^2 \int_S \bar{Q}((\nabla')^2 v)\, dx' - \gamma^2 (1+\varepsilon) \int_S f \cdot \bar{R} \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix} dx' \\ &- \gamma^2 (1+\varepsilon) \int_S f \cdot \bar{R}\bar{W} \begin{pmatrix} 0 \\ \bar{v} \end{pmatrix} dx' - \gamma^2 (1+\varepsilon) \int_S f \cdot \bar{R}\bar{W}^2 \begin{pmatrix} x' \\ 0 \end{pmatrix} dx', \end{aligned}$$

we deduce that

$$\lim_{\gamma \to +\infty} \frac{1}{\gamma^2} J^{\mathrm{VK}}_\varepsilon(\gamma^2 \bar{u}, \gamma \bar{v}, \bar{R}, \gamma \bar{W}) = J^{\mathrm{VK}}_\varepsilon(\bar{u}, \bar{v}, \bar{R}, \bar{W}) < 0.$$

This implies that

$$\lim_{\gamma \to +\infty} J^{\mathrm{VK}}_\varepsilon(\gamma^2 \bar{u}, \gamma \bar{v}, \bar{R}, \gamma \bar{W}) = -\infty,$$

as desired. □

Remark 4.3.5. From the proof it follows that $\inf J^{\mathrm{VK}} = -\infty$ if there is an admissible quadruplet $(\bar{u}, \bar{v}, \bar{R}, \bar{W})$ such that $(\bar{u}, \bar{v}) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ and $J^{\mathrm{VK}}(\bar{u}, \bar{v}, \bar{R}, \bar{W}) < 0$. In this case, one can repeat the same argument with $\varepsilon = 0$.

Remark 4.3.6. We give a short sketch of the proof of Theorem 4.1.5 in the case $\dim \mathcal{R} = 0$. Firstly, we can assume without loss of generality that $\mathcal{R} = \{\mathrm{Id}\}$. Reasoning as in Proposition 2.2.26, since $\mathrm{N}\mathcal{R}_{\mathrm{Id}} = \mathbb{R}^{3\times 3}_{\mathrm{skew}}$, one can show that $ab - c^2 > 0$, where we have written a, b, and c in place of $a(\mathrm{Id}), b(\mathrm{Id})$, and $c(\mathrm{Id})$. Then, arguing as in Proposition 4.3.1, one can prove that, when (S2) holds, any minimizer (u, v, R, W) of J^{VK} with $(u, v) \in \mathcal{B}^{\mathrm{lin}}_{\mathrm{iso}}$ is of the form $(\eta, \delta, \mathrm{Id}, 0)$, with $\eta \in \mathbb{R}^2$ and $\delta \in \mathbb{R}$. Note that, in this setting, stability condition (S2) basically reduces to the *linearized stability* of [LM09] without imposing any additional Dirichlet condition on the boundary. Finally, one can argue as in the proof of Theorem 4.1.5 to conclude.

5 A hierarchy of models for ribbons

5.1 Assumptions and main results

In this chapter, S represents the mid-line of a rod, so that $S := (0, L)$, with $L > 0$. The elastic energy density takes the form $\mathcal{W}(M) := |M^T M - \mathrm{Id}\,|^2$, for $M \in \mathbb{R}^{3\times 2}$. Note that $\mathcal{W}$ satisfies both (RG) and (FI). The total energy is defined as

$$E_h(y) := \int_\Omega \mathcal{W}(\nabla_h y)\,dx + \delta_h^2 \int_\Omega |\nabla_h^2 y|^2\,dx, \qquad y \in W^{2,2}(\Omega;\mathbb{R}^3).$$

Here, $(\delta_h) \subset \mathbb{R}$ is a sequence such that $\delta_h \ll h$ as $h \to 0$. Physically, h and δ_h represent the thickness and the width of a ribbon, respectively. They are both small with respect to the length L, albeit on a different scale.

In this chapter, we study the Γ-convergence of various rescalings of E_h. Precisely, we compute the Γ-limit of

$$E_h^\alpha := \delta_h^{-\alpha} E_h, \qquad \alpha \geq 2.$$

As in the previous chapters, we set $\gamma := \alpha/2$.

In the first part, we treat the Kirchhoff's regime $\alpha = 2$. We prove that, for sequences (y_h) with bounded energy, $\nabla_y y_h \to \begin{pmatrix} d_1 & d_2 \end{pmatrix}$ in $L^2(\Omega, \mathbb{R}^{3\times 2})$, where d_1 and d_2 are independent of x_2 and define a Frenet–Serrin frame for which $\partial_1 d_1 \cdot d_2 = 0$. Since d_1 and d_2 can be thought as being induced by a deformation y of the mid-line S, the constraint $\partial_1 d_1 \cdot d_2 = 0$ is a necessary condition for y being the restriction to S of an isometric immersion of Ω_h, at least for h small enough. We show that E_h^2 Γ-converge to

$$I^{\mathrm{K}}(d_1, d_2) := \int_0^L \left(|\partial_1 d_1|^2 + 2|\partial_2 d_2|^2\right) dx \qquad \text{if } h^{-2}\delta_h \to +\infty,$$

$$I^{\mathrm{S}}(d_1, d_2) := \int_0^L \left(|\partial_1 d_1|^2 + 2|\partial_2 d_2|^2 + Q_s(\partial_1 d_1, \partial_1 d_2)\right) dx \quad \text{if } h^{-2}\delta_h \to 0,$$

where

$$Q_s(u,v) := \min_{\xi\in\mathbb{R}^3} \{|\xi|^2 + 2|u\cdot\xi - v\cdot v|\}. \tag{5.1}$$

The asymptotic behaviour of $h^{-2}\delta_h$ is related to the convergence of $\det(\nabla_h^2 y_h)$, that, roughly speaking, represents the Gauss' curvature. Precisely, when $h^{-2}\delta_h \to 0$, we have $\det(\nabla_h y_h) \overset{*}{\rightharpoonup} 0$ in the sense of measures and the term Q_s emerges as a relaxation of this constraint. The case $\delta_h \sim h^2$ is still open and in the next sections we explain some of the difficulties in the study of this regime.

Then, we move to the so-called Von Kármán's regimes, corresponding to $\alpha > 2$. For a sequence of deformations (y_h) with bounded energy, we identify a displacement $u := (u_1, u_2, u_3)$, where each component is the limit of a suitable rescaling of the displacement at the level h. While u_2 and u_3 are independent of x_2, u_1—the displacement in the x_1-direction—is affine in x_2 and has the form

$$u_1 = \xi - x_2\partial_1 u_2, \tag{5.2}$$

for some $\xi \in W^{1,2}(0,L)$. Moreover, we show that a suitable rescaling of $h^{-1}\partial_2 y_{h,3}$ converges to a function θ, depending only on x_1 and representing the twist of the ribbon.

We prove that E_h^α Γ-converge to

$$I^{\mathrm{VK}}(u_1,u_3,\theta) := \int_0^L \left(|\partial_{11}u_3|^2 + 2|\partial_1\theta|^2\right) dx + \int_\Omega |(\partial_1 u_3)^2 + 2\partial_1 u_1|^2\, dx \quad \text{if } \alpha = 4,$$

$$I^{\mathrm{LVK}}(u_1,u_3,\theta) := \int_0^L \left(|\partial_{11}u_3|^2 + 2|\partial_1\theta|^2\right) dx + 4\int_\Omega |\partial_1 u_1|^2\, dx \quad \text{if } \alpha > 4.$$

Note that both I^{VK} and I^{LVK} can be equivalently rewritten in terms of ξ, u_2, u_3, and θ.

As in Chapters 3 and 4, when $2 < \alpha < 4$ some constraint relating the in-plane and out-of-plane displacement appears at the limit. This is also the case here, where we have that

$$(\partial_1 u_3)^2 + 2\partial_1 u_1 = 0. \tag{5.3}$$

Note that, by (5.2)–(5.3) it follows that $\partial_1 u_2 = 0$ and u_1 is independent of x_2. Differently from the analogous constraint for plates, given $u_3 \in W^{2,2}(0,L)$, there always exists u_1 such that (5.3) is satisfied.

When $2 < \alpha < 4$ and $h^{-2}\delta_h^{2-\gamma} \to +\infty$, we show that E_h^α Γ-converge to

$$I^{\mathrm{CVK}}(u_3,\theta) := \int_0^L \left(|\partial_{11}u_3|^2 + 2|\partial_1\theta|^2\right) dx.$$

Instead, when $2 < \alpha < 4$ and $h^{-2}\delta_h^{2-\gamma} \to 0$, we have the Γ-convergence to

$$I^{\mathrm{VKS}}(u_3,\theta) := \int_0^L \left(|\partial_{11}u_3|^2 + 2|\partial_1\theta|^2 + Q_s(\partial_{11}u_3, \partial_1\theta)\right) dx,$$

where, with some abuse of notation we still denote by Q_s the function

$$Q_s(a,b) := \min_{c\in\mathbb{R}} \{c^2 + 2|ac - b^2|\}. \tag{5.4}$$

As for the Kirchhoff's case, the behaviour of $h^{-2}\delta_h^{2-\gamma}$ is linked to the convergence of a suitable rescaling of the Gauss' curvature. The case $\delta_h^{2-\gamma} \sim h^2$ is still open and presents analogous difficulties to the case $\alpha = 2$ and $\delta_h \sim h^2$.

5.2 The Kirchhoff's regime

In this section we first prove the compactness and then move to the Γ-convergence result for $\alpha = 2$.

5.2.1 Compactness

Theorem 5.2.1. *Let $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ be a sequence such that $E_h^2(y_h) \leq C$. Then, there are functions $y \in W^{2,2}(0,L;\mathbb{R}^3)$, $d_2 \in W^{1,2}(0,L;\mathbb{R}^3)$ and $\rho \in L^2(\Omega;\mathbb{R}^3)$ such that, up to a subsequence and to translation,*

(i) $y_h \rightharpoonup y$ *in* $W^{2,2}(\Omega;\mathbb{R}^3)$,

(ii) $h^{-1}\partial_2 y_h \rightharpoonup d_2$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$,

(iii) $h^{-2}\partial_2^2 y_h \rightharpoonup \rho$ *in* $L^2(\Omega;\mathbb{R}^3)$.

Moreover, $d_1 := \partial_1 y$ and d_2 are unitary and orthogonal almost everywhere. Lastly, $\partial_1 d_1$, $\partial_1 d_2$, and ρ are almost everywhere parallel to $d_3 := d_1 \wedge d_2$.

Proof. By the boundedness of the energy, we immediately deduce (up to a subsequence) the weak convergences of the rescaled gradient and Hessian

(a) $\nabla_h^2 y_h \rightharpoonup \mathbb{B}$ in $L^2(\Omega;\mathbb{R}^{3\times2\times2})$,

(b) $\nabla_h y_h \rightharpoonup A$ in $L^4(\Omega;\mathbb{R}^{3\times2})$.

Indeed, the rescaled gradient is bounded in L^4 given that

$$|M^TM| \geq \frac{1}{\sqrt{2}}|M|^2 \qquad \forall\, M \in \mathbb{R}^{3\times2}.$$

We call d_1 and d_2 the columns of A. Up to translation, we have that $y_h \rightharpoonup y$ in $W^{2,2}(\Omega;\mathbb{R}^3)$. Clearly, since $\partial_2 y_h \to 0$ in $W^{1,2}(\Omega;\mathbb{R}^3)$, the function y is independent of x_2. Moreover, $\nabla_h y_h \rightharpoonup A$ in $W^{1,2}(\Omega;\mathbb{R}^{3\times2})$ so that the convergence $\nabla_h y_h \to A$ is strong in $L^4(\Omega;\mathbb{R}^{3\times2})$. This, combined with the strong convergence $\nabla_h y_h^T \nabla_h y_h \to \mathrm{Id}$ in $L^2(\Omega;\mathbb{R}^{2\times2})$ following by the energy, allow us to deduce that $A^TA = \mathrm{Id}$ so that d_1 and d_2 are unitary and orthogonal almost everywhere. Since $h^{-1}\partial_2^2 y_h \to 0$ strongly in $L^2(\Omega;\mathbb{R}^3)$ we get $\partial_2 d_2 = 0$, i.e., d_2 is independent of x_2.

To simplify the notation, we will write the tensor $\mathbb{B}$ as a 2×2 matrix whose entries are $\mathbb{R}^3$ vectors. We have already deduced that $\mathbb{B}_{11} = \partial_1^2 y = \partial_1 d_1$ and that $\mathbb{B}_{12} = \mathbb{B}_{21} = \partial_1 d_2$. Denote with ρ the vector $\mathbb{B}_{22}$. We now show that $\partial_1 d_1$, $\partial_1 d_2$, and ρ are orthogonal to both d_1 and d_2. Observe that

$$\frac{1}{h}\|\partial_2(\nabla_h y_h^T\nabla_h y_h - \mathrm{Id})\|_{(W^{1,2})^*} \leq C\frac{1}{h}\|\nabla_h y_h^T\nabla_h y_h - \mathrm{Id}\,\|_{L^2} \leq C\frac{\delta_h}{h} \to 0.$$

Moreover,

$$\begin{aligned}&\frac{1}{h}\partial_2(\nabla_h y_h^T\nabla_h y_h - \mathrm{Id})\\ &\quad= \frac{1}{h}\left[\begin{pmatrix}(\partial_{12}y_h)^T\\ h^{-1}(\partial_{22}y_h)^T\end{pmatrix}\begin{pmatrix}\partial_1 y_h & h^{-1}\partial_2 y_h\end{pmatrix} + \begin{pmatrix}(\partial_1 y_h)^T\\ h^{-1}(\partial_2 y_h)^T\end{pmatrix}\begin{pmatrix}\partial_{12}y_h & h^{-1}\partial_{22}y_h\end{pmatrix}\right].\end{aligned}$$

Passing to the limit we deduce that

$$\begin{pmatrix} (\partial_1 d_2)^T \\ \rho^T \end{pmatrix} \begin{pmatrix} d_1 & d_2 \end{pmatrix} + \begin{pmatrix} d_1^T \\ d_2^T \end{pmatrix} \begin{pmatrix} \partial_1 d_2 & \rho \end{pmatrix} = 0. \tag{5.5}$$

Componentwise, (5.5) gives

$$\begin{aligned} &2\partial_1 d_2 \cdot d_1 = 0, \\ &\partial_1 d_2 \cdot d_2 + d_1 \cdot \rho = 0, \\ &2\rho \cdot d_2 = 0. \end{aligned}$$

Hence, $\partial_1 d_2$ is orthogonal to d_1 and ρ is orthogonal to d_2. Differentiating $|d_2|^2 = 1$, we deduce that $\partial_1 d_2 \cdot d_2 = 0$, that gives both the desired result for both d_2 and ρ. Differentiating the identity $d_1 \cdot d_2 = 0$ and $d_1 \cdot d_1 = 1$ we get

$$\begin{aligned} &2d_1 \cdot \partial_1 d_1 = 0, \\ &\partial_1 d_1 \cdot d_2 + d_1 \cdot \partial_1 d_2 = 0, \end{aligned}$$

which gives the orthogonality of $\partial_1 d_1$ to both d_1 and d_2. □

The triplet (d_1, d_2, d_3) is a Frenet–Serret frame representing a deformed configuration of the mid-line. The extra condition that $\partial_1 d_1$ and ∂d_2 are both parallel to d_3 ensures that this frame arises as the restriction on the mid-line of an isometry of a thin strip.

To simplify the notation we introduce the set

$$\mathcal{F}_K \subset W^{2,2}(0, L; \mathbb{R}^3) \times W^{1,2}(0, L; \mathbb{R}^3),$$

representing the admissible Frenet–Serret frames. Precisely, we say that $(y, d_2) \in \mathcal{F}_K$ if, setting $d_1 := \partial_1 y$,

(i) $d_1 \cdot d_1 = d_2 \cdot d_2 = 1$ a.e. in $(0, L)$,

(ii) $d_1 \cdot d_2 = 0$ a.e. in $(0, L)$,

(iii) $\partial_1 d_1 \cdot d_2 = \partial_1 d_2 \cdot d_1 = 0$ a.e. in $(0, L)$.

Moreover, given a sequence of deformations $(y_h) \subset W^{2,2}(\Omega; \mathbb{R}^3)$ and a couple $(y, d_2) \in \mathcal{F}_K$, we say that $y_h \xrightarrow{\mathcal{F}_K} (y, d_2)$ if

(i) $y_h \rightharpoonup y$ in $W^{2,2}(\Omega; \mathbb{R}^3)$,

(ii) $h^{-1}\partial_2 y_h \rightharpoonup d_2$ in $W^{1,2}(\Omega; \mathbb{R}^3)$.

We compute the Γ-convergence of E_h^2 with respect to the above convergence, that is the natural one emerging from the boundedness of the energy.

5.2.2 Γ-convergence for $h^2 \ll \delta_h \ll h$

Theorem 5.2.2. *Suppose that $h^2 \ll \delta_h \ll h$. Then for any sequence of deformations $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that $y_h \xrightarrow{\mathcal{F}_K} (y, d_2)$ for some $(y, d_2) \in \mathcal{F}_K$ we have*

$$\liminf_{h\to 0^+} E_h^2(y_h) \geq I^K(d_1, d_2), \tag{5.6}$$

where $d_1 := \partial_1 y$. Moreover, for any choice of $(y, d_2) \in \mathcal{F}_K$ there is a sequence of deformations $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that $y_h \xrightarrow{\mathcal{F}_K} (y, d_2)$ and

$$\lim_{h\to 0^+} E_h^2(y_h) = I^K(d_1, d_2),$$

where $d_1 := \partial_1 y$.

Proof. We simply have

$$E_h^2(y_h) \geq \int_\Omega |\nabla_h^2 y_h|^2 \, dx \geq \int_\Omega \left(|\partial_{11} y_h|^2 + 2|\partial_{12} y_h|^2\right) \, dx.$$

Passing to the limit we deduce (5.6) by lower semicontinuity. Let $(y, d_2) \in \mathcal{F}_K \cap (C^\infty([0, L];\mathbb{R}^3))^2$. We define

$$y_h := y + hx_2 d_2.$$

Clearly, $y_h \rightharpoonup y$ in $W^{2,2}(\Omega;\mathbb{R}^3)$. We get

$$\nabla_h y_h = \begin{pmatrix} d_1 & d_2 \end{pmatrix} + hx_2 \begin{pmatrix} \partial_1 d_2 & 0 \end{pmatrix},$$

where $d_1 := \partial_1 y$. In particular, $h^{-1}\partial_2 y_h \rightharpoonup d_2$, showing that $y_h \xrightarrow{\mathcal{F}_K} (y, d_2)$. Computing the elastic energy density term we get

$$\nabla_h^T y_h \nabla_h y_h - \mathrm{Id} = O(h^2).$$

Differentiating again we obtain

$$\nabla_h^2 y_h = \begin{pmatrix} \partial_1 d_1 & \partial_1 d_2 \\ \partial_1 d_2 & 0 \end{pmatrix} + o(h).$$

Thus, since $h^4/\delta_h^2 \to 0$, passing to the limit in the total energy we get

$$E_h^2(y_h) \to \int_0^L |\partial_1 d_1|^2 \, dx + 2\int_0^L |\partial_1 d_2|^2 \, dx = I^K(d_1, d_2).$$

To conclude for a general pair $(y, d_2) \in \mathcal{F}_K$ we shall apply a density argument. It is enough to prove that for every couple $(y, d_2) \in \mathcal{F}_K$ there is a sequence

$$(y_n, d_2^n) \subset (C^\infty([0, L];\mathbb{R}^3))^2 \cap \mathcal{F}_K$$

such that $y_n \to y$ in $W^{2,2}(0, L;\mathbb{R}^3)$ and $d_2^n \to d_2$ in $W^{1,2}(0, L;\mathbb{R}^3)$. Let $(y, d_2) \in \mathcal{F}_K$. Define $d_1 := \partial_1 y$, $\kappa := \partial_1 d_1 \cdot d_3$, and $\tau := \partial_1 d_2 \cdot d_3$, where $d_3 := d_1 \wedge d_2$. By density, there exist two sequences

$$(\tau_n),\ (\kappa_n) \subset C^\infty([0, L])$$

such that $\tau_n \to \tau$ and $\kappa_n \to \kappa$ in $L^2(0, L)$. Let

$$A_n := \begin{pmatrix} 0 & 0 & -\kappa_n \\ 0 & 0 & -\tau_n \\ \kappa_n & \tau_n & 0 \end{pmatrix}.$$

Denote by $R_n \in C^\infty([0, L])$ the unique global solution of the Cauchy's problem

$$\begin{cases} \partial_1 X = XA_n & \text{in } [0, L], \\ X(0) = \begin{pmatrix} d_1(0) & d_2(0) & d_3(0) \end{pmatrix}. \end{cases} \tag{5.7}$$

Note that $R_n(0) \in SO(3)$ and that

$$\partial_1(R_n R_n^T) = \partial_1 R_n R_n^T + R_n(\partial_1 R_n)^T = (R_n A_n R_n^T + R_n A_n^T R_n^T) = 0,$$

that is $R_n \in SO(3)$ everywhere in $[0, L]$. We define

$$y_n(t) := y(0) + \int_0^t R_n(s)e_1\, ds,$$
$$d_2^n(t) := R_n(t)e_2.$$

By construction, we have that

$$(y_n, d_2^n) \in (C^\infty(0, L; \mathbb{R}^3))^2 \cap \mathcal{F}_K.$$

Indeed, defining $d_1^n := \partial_1 y_n$, we get

$$\partial_1 d_1^n \cdot d_2^n = \partial_1 R_n e_1 \cdot R_n e_2 = \kappa_n R_n e_1 \cdot R_n e_2 = 0,$$
$$d_1^n \cdot \partial_1 d_2^n = R_n e_1 \cdot \partial_1 R_n e_2 = \tau_n R_n e_1 \cdot R_n e_2 = 0.$$

Clearly, up to subsequences, $R_n \rightharpoonup R$ in $W^{1,2}(0, L; \mathbb{R}^{3\times 3})$. Moreover, passing to the limit in (5.7), we deduce that R is the unique solution of

$$\begin{cases} \partial_1 X = XA & \text{in } [0, L], \\ X(0) = \begin{pmatrix} d_1(0) & d_2(0) & d_3(0) \end{pmatrix}. \end{cases} \tag{5.8}$$

with

$$A := \begin{pmatrix} 0 & 0 & -\kappa \\ 0 & 0 & -\tau \\ \kappa & \tau & 0 \end{pmatrix}.$$

In particular, $R_n \to R$ in $W^{1,2}(0, L; \mathbb{R}^3 \times 3)$. Since the matrix

$$\begin{pmatrix} d_1 & d_2 & d_3 \end{pmatrix}$$

is a solution of (5.8), we conclude that

$$R = \begin{pmatrix} d_1 & d_2 & d_3 \end{pmatrix},$$

giving the convergence of d_2^n. The convergence of y_n follows from its definition. $\square$

5.2.3 Γ-convergence for $\delta_h \ll h^2$

When $\delta_h \ll h^2$, we cannot use the same recovery sequence of the proof of Theorem 5.2.2. Indeed, the term involving the elastic energy density $\mathcal{W}$ would blow up. The main difference is that, as we show now, the Gauss' curvature converge to 0 in the sense of measures.

Lemma 5.2.3. *Suppose $\delta_h \ll h^2$ and let $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations such that $E_h^2(y_h) \leq C$. Then*

$$\det(\nabla_h^2 y_h) \overset{*}{\rightharpoonup} 0 \quad in \quad \mathcal{M}_b(\Omega).$$

Proof. Let $G^h := \nabla_h y_h^T \nabla_h y_h - \mathrm{Id}$. Observe that, in the sense of distributions, it holds that

$$\partial_{11} G_{22}^h + \frac{1}{h^2}\partial_{22} G_{11}^h + \frac{2}{h}\partial_{12} G_{12}^h = -2\det(\nabla_h^2 y_h). \tag{5.9}$$

Given that $\|G_h\|_{L^2} \leq C\delta_h \ll Ch^2$ we deduce that $\det(\nabla_h^2 y_h) \to 0$ in the sense of distributions. By the energy, we can also bound the L^1 norm of the determinant so that, up to a subsequence, we have the weak-star convergence $\det(\nabla_h^2 y_h) \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_b(\Omega)$. By the uniqueness of the limit, we conclude that $\mu = 0$, finishing the proof. □

To construct the recovery sequences, we resort to a careful construction of isometries y_h of the thin strip Ω_h, for which $\det(\nabla_h^2 y_h) \equiv 0$ and to a relaxation argument. We start with a simple observation regarding the norm of the Hessian of an isometric immersion and a preliminary lemma that justifies the ambiguous notation for Q_s (see (5.1) and (5.4)).

Proposition 5.2.4. *Let $u \in W^{2,2}_{\mathrm{iso}}(\Omega;\mathbb{R}^3)$. Then*

$$|\nabla^2 u|^2 = \sum_{i,j=1}^{2} |\partial_{ij} u \cdot \nu|^2,$$

where $\nu := \partial_1 u \wedge \partial_2 u$.

Proof. Differentiating the identity $\nabla u^T \nabla u = \mathrm{Id}$, we deduce that

$$\begin{aligned}
&\partial_{11} u \cdot \partial_1 u = 0,\\
&\partial_{12} u \cdot \partial_1 u = 0,\\
&\partial_{11} u \cdot \partial_2 u + \partial_{12} u \cdot \partial_1 u = 0,\\
&\partial_{12} u \cdot \partial_2 u + \partial_{22} u \cdot \partial_1 u = 0,\\
&\partial_{12} u \cdot \partial_2 u = 0,\\
&\partial_{22} u \cdot \partial_2 u = 0.
\end{aligned}$$

It easily follows that $\partial_{ij} u$ is parallel to ν for every $i, j = 1, 2$, and the proof is concluded. □

Lemma 5.2.5. *Let* $(y, d_2) \in \mathcal{F}_K$ *and define* $d_1 := \partial_1 y$. *Let* $\kappa := \partial_1 d_1 \cdot d_3$ *and* $\tau := \partial_1 d_2 \cdot d_3$, *where* $d_3 := d_1 \wedge d_2$. *Then*

$$Q_s(\partial_1 d_1, \partial_1 d_2) = Q_s(\kappa, \tau),$$

where we tacitly passed from Q_s *defined as in* (5.1) *to* Q_s *defined as in* (5.4).

Proof. Since both $\partial_1 d_1$ and $\partial_1 d_2$ are parallel to d_3, and d_3 is unitary, we have $|\partial_1 d_1| = |\kappa|$ and $|\partial_1 d_2| = |\tau|$. In particular, for $\xi \in \mathbb{R}^3$,

$$|\xi|^2 + 2|\partial_1 d_1 \cdot \xi - \partial_1 d_2 \cdot \partial_1 d_2| = |\xi \cdot d_1|^2 + |\xi \cdot d_2|^2 + |\xi \cdot d_3|^2 + 2|\kappa\xi \cdot d_3 - \tau^2|.$$

Then, if $\bar{\xi}$ is the minimizer of

$$\xi \mapsto |\xi|^2 + 2|\partial_1 d_1 \cdot \xi - \partial_1 d_2 \cdot \partial_1 d_2|,$$

it must be such that $\bar{\xi} \cdot d_1 = \bar{\xi} \cdot d_2 = 0$, and $\bar{\xi} \cdot d_3$ minimizes

$$c \mapsto c^2 + 2|\kappa c - \tau^2|,$$

concluding the proof. □

As we anticipated, the construction of the recovery sequence is based on a relaxation argument. The next Lemma is thus concerned with the lower semicontinuous envelope of the map

$$\Psi : L^2(0, L; \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \to \mathbb{R}, \qquad \Psi(M) := \begin{cases} \int_0^L |M|^2\,dx & \text{if } \det(M) = 0 \text{ a.e.,} \\ +\infty & \text{otherwise,} \end{cases} \tag{5.10}$$

with respect to the weak L^2 topology (see also Remark 5.2.8).

Lemma 5.2.6. *For every matrix-valued field* $M \in L^2(0, L; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ *there exists a sequence*

$$(M_n) \subset L^2(0, L; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

such that for every $n \in \mathbb{N}$ *we have* $\det(M_n) = 0$ *almost everywhere on* $(0, L)$, $M_n \rightharpoonup M$ *in* L^2, *and*

$$\int_0^L |M_n|^2\,dx \to \int_0^L |M|^2\,dx + 2\int_0^L |\det(M)|\,dx.$$

Moreover, the sequence M_n *can be choosen of the form*

$$M_n = \lambda_n p_n \otimes p_n$$

for some $\lambda_n \in C^\infty([0, L])$ *and* $p_n \in C^\infty([0, L]; \mathbb{R}^2)$ *with* $|p_n| \equiv 1$ *and* $p_n \neq e_2$ *on* $[0, L]$.

Proof. In this proof, we follow [Fre+15, Lemma 3.1] and [Fre+16, Lemma 16].
Step 1. Suppose first that M is constant and diagonal. If $\det(M) = 0$, the result is trivial, thus we may assume that $\det(M) \neq 0$. Let

$$\theta := \frac{|M_{11}|}{|M_{11}| + |M_{22}|}.$$

Since M has nonzero determinant, $\theta \in (0,1)$. Moreover

$$\frac{M_{11}^2}{\theta} + \frac{M_{22}^2}{1-\theta} = |M_{11}|^2 + |M_{22}|^2 + 2|M_{11}M_{22}| = |M|^2 + 2|\det(M)|. \tag{5.11}$$

Let $\chi(x) := \chi_{(0,\theta)}(x - \lfloor x \rfloor)$. Now define $\chi_n \in L^\infty(0,L)$ as $\chi_n(x_1) := \chi(nx_1)$. Since χ is 1-periodic with average over one period given by θ, we have $\chi_n \overset{*}{\rightharpoonup} \theta$ in $L^\infty(0,L)$. In particular, defining

$$M_n(x_1) := \chi_n(x_1)\frac{M_{11}}{\theta} e_1 \otimes e_1 + (1 - \chi_n(x_1))\frac{M_{22}}{1-\theta} e_2 \otimes e_2,$$

we get $\det(M_n) \equiv 0$, $M_n \rightharpoonup M$ in L^2, and by (5.11)

$$\begin{aligned}\int_0^L |M_n|^2\,dx &= \int_0^L \left(\chi_n(x_1)\frac{M_{11}^2}{\theta^2} + (1-\chi_n(x_1))\frac{M_{22}^2}{(1-\theta)^2}\right) dx \\ &\to \int_0^L |M|^2\,dx + 2\int_0^L |\det(M)|\,dx.\end{aligned}$$

If M is constant, but non diagonal, it can be diagonalized by an orthogonal matrix $Q \in \mathrm{O}(2)$, namely $Q^T M Q$ is diagonal. Construct a sequence $\widetilde{M}_n$ as above for $Q^T M Q$, and define $M_n := Q\widetilde{M}_n Q^T$. We have that $M_n \rightharpoonup M$ in L^2 and

$$\begin{aligned}\int_0^L |M_n|^2\,dx = \int_0^L |\widetilde{M}_n|^2\,dx &\to \int_0^L |Q^T M Q|^2\,dx + 2\int_0^L |\det(Q^T M Q)|\,dx \\ &= \int_0^L |M|^2\,dx + 2\int_0^L |\det(M)|\,dx.\end{aligned}$$

Step 2. Suppose now that M is piecewise constant. Then, gluing together the construction of Step 1 on each set where M is constant, we get the desired sequence.
Step 3. For an arbitrary $M \in L^2(0,L;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ there is a sequence

$$(M_k) \subset L^2(0,L;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

of piecewise constant functions such that $M_k \to M$ in L^2. For every $k \in \mathbb{N}$ we apply Step 2, and construct sequences

$$(M_{n,k}) \subset L^2(0,L;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

such that $\det(M_{n,k}) = 0$ almost everywhere, $M_{n,k} \rightharpoonup M_k$ in L^2 as $n \to \infty$, and

$$\int_0^L |M_{n,k}|^2\,dx \xrightarrow{n\to\infty} \int_0^L |M_k|^2\,dx + 2\int_0^L |\det(M_k)|\,dx. \tag{5.12}$$

By (5.12), the matrix-valued fields $M_{n,k}$ can be chosen uniformly bounded—with respect to both n and k—in L^2. Then, since the weak topology of L^2 is metrizable on balls, we conclude by a diagonal argument.

We are left to prove that the sequence we have constructed can be chosen of the form

$$M_n = \lambda_n p_n \otimes p_n$$

for some $\lambda_n \in C^\infty([0,L])$ and $p_n \in C^\infty([0,L];\mathbb{R}^2)$ with $|p_n| \equiv 1$ and $p_n \neq e_2$ on $[0,L]$.

Up to a further diagonal argument, it is enough to show that for every matrix-valued field $M \in L^2(0,L;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with $\det(M) \neq 0$ almost everywhere on $[0,L]$ there are sequences $(\lambda_n) \subset C^\infty([0,L])$ and $(p_n) \subset C^\infty([0,L];\mathbb{R}^2)$ with $|p_n| \equiv 1$ and $p_n \neq e_2$ everywhere on $[0,L]$ such that

$$\lambda_n p_n \otimes p_n \to M \qquad \text{in } L^2(0,L;\mathbb{R}^{2\times 2}_{\mathrm{sym}}).$$

Thus, let $M \in L^2(0,L;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ be such that $\det(M) = 0$ almost everywhere. Firstly, since M is symmetric, we have $M = \lambda p \otimes p$, where $\lambda := \mathrm{Tr}(M)$ and

$$p := \begin{cases} \mathrm{sign}(M_{11}) Me_1/|Me_1| & \text{if } Me_1 \neq 0, \\ e_2 & \text{if } Me_1 = 0 \end{cases}$$

Indeed, if $Me_1 = 0$, then by symmetry

$$M = \begin{pmatrix} 0 & 0 \\ 0 & M_{22} \end{pmatrix} = M_{22} e_2 \otimes e_2$$

and $\mathrm{Tr}(M) = M_{22}$. If instead $Me_1 \neq 0$, we have

$$\begin{aligned}
(p\otimes p)_{11} &= \frac{M_{11}^2}{|Me_1|^2} = M_{11}\frac{M_{11}}{|Me_1|^2}, \\
(p\otimes p)_{12} &= \frac{M_{11}M_{12}}{|Me_1|^2} = M_{12}\frac{M_{11}}{|Me_1|^2}, \\
(p\otimes p)_{22} &= \frac{M_{12}^2}{|Me_1|^2} = M_{22}\frac{M_{11}}{|Me_1|^2},
\end{aligned}$$

and

$$\mathrm{Tr}(M) = M_{11} + M_{22} = M_{11} + \frac{M_{12}^2}{M_{11}} = \frac{|Me_1|^2}{M_{11}}.$$

Since $p_1 \geq 0$, the unit vector p can be expressed by means of a map

$$\beta \in L^\infty\Big(0,L;\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big]\Big)$$

as

$$p = \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix}.$$

Define

$$\widetilde{\beta}_n := \Big(\frac{1}{n} - \frac{\pi}{2}\Big) \vee \Big(\beta \wedge \Big(\frac{\pi}{2} - \frac{1}{n}\Big)\Big).$$

and $\beta_n := \rho_n * \widetilde{\beta}_n$, where ρ_n is a standard mollifier with support contained in the ball of radius $1/n$ and, with a little abuse of notation, we still denote with $\widetilde{\beta}_n$ its extension by zero to $\mathbb{R}$. Clearly, $\beta_n \in C^\infty([0,L])$ and $\beta_n \neq \pi/2$ on $[0,L]$. Moreover

$$\begin{aligned}
|\beta_n(x) - \beta(x)| &\leq \int_{\mathbb{R}} |(\widetilde{\beta}_n(y) - \beta(x))\rho_n(x-y)|\,dy \leq \int_{-\frac{1}{n}}^{\frac{1}{n}} |\widetilde{\beta}_n(x-y) - \beta(x)|\,dy \\
&\leq 2\frac{1}{n^2} \to 0
\end{aligned}$$

Thus, $\beta_n \to \beta$ pointwise, and by Dominated Convergence Theorem, for every $q > 1$

$$p_n := \begin{pmatrix} \cos(\beta_n) \\ \sin(\beta_n) \end{pmatrix} \to p \qquad \text{in } L^q(0, L; \mathbb{R}^2).$$

Clearly, $p_n \in C^\infty([0, L]; \mathbb{R}^2)$, and $p_n \neq e_2$ everywhere on $[0, L]$. Let $(\lambda_n) \subset C^\infty([0, L])$ be a sequence such that $\lambda_n \to \lambda$ in $L^2(0, L)$. Defining

$$M_n := \lambda_n p_n \otimes p_n.$$

we conclude. □

We are ready to prove the Γ-convergence result.

Theorem 5.2.7. *Suppose that $\delta_h \ll h^2$. Then for any sequence of deformations $(y_h) \subset W^{2,2}(\Omega; \mathbb{R}^3)$ such that $y_h \xrightarrow{\mathcal{F}_\mathrm{K}} (y, d_2)$ for some $(y, d_2) \in \mathcal{F}_\mathrm{K}$ it holds*

$$\liminf_{h \to 0^+} E_h^2(y_h) \geq I^\mathrm{S}(d_1, d_2), \tag{5.13}$$

where $d_1 := \partial_1 y$. Moreover, for any choice of $(y, d_2) \in \mathcal{F}_\mathrm{K}$ there is a sequence of deformations $(y_h) \subset W^{2,2}(\Omega; \mathbb{R}^3)$ such that $y_h \xrightarrow{\mathcal{F}_\mathrm{K}} (y, d_2)$ and

$$\lim_{h \to 0^+} E_h^2(y_h) = I^\mathrm{S}(d_1, d_2),$$

where $d_1 := \partial_1 y$.

Proof. We start by proving the liminf inequality (5.13). Without loss of generality we can suppose that $E_h^2(y_h) \leq C$. For every $\varphi \in C_0^1(\Omega; [0, 1])$ we get

$$\begin{aligned} E_h^2(y_h) \geq \int_\Omega |\nabla_h^2 y_h|^2 \, dx &= \int_\Omega \varphi \big(|\partial_{11} y_h|^2 + 2|h^{-1}\partial_{12} y_h|^2 + |h^{-2}\partial_{22} y_h|^2 \big) \, dx \\ &\quad + \int_\Omega (1 - \varphi) \big(|\partial_{11} y_h|^2 + 2|h^{-1}\partial_{12} y_h|^2 + |h^{-2}\partial_{22} y_h|^2 \big) \, dx \\ &= \int_\Omega \varphi |\partial_{11} y_h + h^{-2}\partial_{22} y_h|^2 \, dx - 2 \int_\Omega \varphi \det(\nabla_h^2 y_h) \, dx \\ &\quad + \int_\Omega (1 - \varphi) \big(|\partial_{11} y_h - h^{-2}\partial_{22} y_h|^2 + 4|h^{-1}\partial_{12} y_h|^2 \big) \, dx \\ &\quad + 2 \int_\Omega (1 - \varphi) \det(\nabla_h^2 y_h) \, dx. \end{aligned}$$

Since φ is non-negative, all the terms are lower semicontinuous with respect to the weak convergences in L^2 by convexity. By Lemma 5.2.3 and Theorem 5.2.1, passing to the limit we deduce that

$$\begin{aligned} \liminf_{h \to 0^+} E_h^2(y_h) &\geq \int_\Omega \varphi |\partial_1 d_1 + \rho|^2 \, dx + \int_\Omega (1 - \varphi) \big(|\partial_1 d_1 - \rho|^2 + 4|\partial_1 d_2|^2 \big) \, dx \\ &= \int_\Omega |\mathbb{B}|^2 \, dx + 2 \int_\Omega \det(\mathbb{B}) \, dx - 2 \int_\Omega \det(\mathbb{B}) \, dx, \end{aligned} \tag{5.14}$$

where

$$\mathbb{B} := \begin{pmatrix} \partial_1 d_1 & \partial_1 d_2 \\ \partial_1 d_2 & \rho \end{pmatrix}.$$

Let

$$\Omega^+ := \{x \in \Omega\colon\ \det(\mathbb{B}(x)) \geq 0\}$$

and $\Omega^- := \Omega \backslash \Omega^+$. Pick a sequence $(\varphi_n) \subset C_0^1(\Omega; [0,1])$ such that $\varphi_n \to \chi_{\Omega^+}$ almost everywhere. By Dominated Convergence Theorem, from (5.14) we deduce that

$$\begin{aligned}
\liminf_{h\to 0^+} E_h^2(y_h) &\geq \int_\Omega |\mathbb{B}|^2\, dx + 2\int_{\Omega^+} \varphi \det(\mathbb{B})\, dx - 2\int_{\Omega^-} (1-\varphi)\det(\mathbb{B})\, dx \\
&= \int_\Omega (|\mathbb{B}|^2 + 2|\det(\mathbb{B})|)\, dx \\
&\geq \int_0^L \left(|\partial_1 d_1|^2 + 2|\partial_1 d_2|^2 + Q_s(\partial_1 d_1, \partial_1 d_2)\right) dx = I^{\mathrm{S}}(d_1, d_2).
\end{aligned}$$

We move now to the recovery sequence construction. Let

$$(y, d_2) \in \mathcal{F}_{\mathrm{K}} \cap (C^\infty([0,L]; \mathbb{R}^3))^2.$$

Define $\kappa := \partial_1 d_1 \cdot d_3$ and $\tau := \partial_1 d_2 \cdot d_3$, where $d_1 := \partial_1 y$ and $d_3 := d_1 \wedge d_2$. For $x_1 \in (0,L)$, let $c(x_1) \in \mathbb{R}$ be such that

$$|c(x_1)|^2 + 2|\kappa(x_1)c(x_1) - \tau^2(x_1)| = Q_s(\kappa(x), \tau(x)),$$

where Q_s is defined as in (5.4). It is a simple computation to prove that

$$c(x_1) = \begin{cases} \tau^2(x_1)/\kappa(x_1) & \text{if } |\kappa(x_1)| > |\tau(x_1)|, \\ \kappa(x_1) & \text{if } |\kappa(x_1)| \leq |\tau(x_1)|. \end{cases}$$

Thus, $c \in L^2(0,L)$. Consider the matrix-valued field

$$M := \begin{pmatrix} \kappa & \tau \\ \tau & c \end{pmatrix}.$$

By Lemma 5.2.6 there exist sequences $(\lambda^n) \subset C^\infty([0,L])$ and $(p^n) \subset C^\infty([0,L]; \mathbb{R}^2)$ with $|p^n| \equiv 1$, $p^n \neq e_2$ everywhere on $[0,L]$, such that, defining

$$M^n := \lambda^n p^n \otimes p^n,$$

we have $M^n \rightharpoonup M$ and

$$\int_0^L |M^n|^2\, dx \to \int_0^L |M|^2\, dx + 2\int_0^L |\det(M)|\, dx = \int_0^L (|\kappa|^2 + 2|\tau|^2 + Q_s(\kappa, \tau))\, dx$$

Note that, since both $\partial_1 d_1$ and $\partial_1 d_2$ are parallel to d_3 and d_3 is unitary, we have $|\kappa| = |\partial_1 d_1|$ and $|\tau| = |\partial_1 d_2|$. By Lemma 5.2.5

$$Q_s(\partial_1 d_1, \partial_1 d_2) = Q_s(\kappa, \tau),$$

where we tacitly passed from Q_s defined as in (5.1) to Q_s defined as in (5.4). It follows that

$$\int_0^L |M^n|^2\, dx \to I^{\mathrm{S}}(d_1, d_2). \tag{5.15}$$

Denote by $R^n \in C^\infty([0,L])$ the unique global solution of the Cauchy's problem

$$\begin{cases} \partial_1 X = XA^n & \text{in } [0,L], \\ X(0) = \begin{pmatrix} d_1(0) & d_2(0) & d_3(0) \end{pmatrix}, \end{cases}$$

where

$$A^n := \begin{pmatrix} 0 & 0 & -(M^n)_{11} \\ 0 & 0 & -(M^n)_{12} \\ (M^n)_{11} & (M^n)_{12} & 0 \end{pmatrix}.$$

Define $d_1^n := R^n e_1$, and $d_2^n := R^n e_2$. Let

$$y^n(x_1) = y(0) + \int_0^{x_1} d_1^n(s)\, ds.$$

Arguing as in the proof of Theorem 5.2.2, $R^n \in \mathrm{SO}(3)$ everywhere on $[0,L]$ and so $(y^n, d_2^n) \in \mathcal{F}_{\mathrm{K}}$.

We are now in the position to apply Theorem 2.2.15 with $p := p^n$, $y := y^n$, $d_2 := d_2^n$, and $\lambda := \lambda^n$, up to a continuous extension on a neighbourhood of $[0,L]$. Thus, for some $\varepsilon^n > 0$, there exists, an isometry

$$u^n \in W^{2,2}_{\mathrm{iso}}(\Omega_{\varepsilon^n}; \mathbb{R}^3) \cap W^{1,\infty}(\Omega_{\varepsilon^n}; \mathbb{R}^3)$$

satisfying (a)–(c) of Theorem 2.2.15. For $h \le \varepsilon^n$, we define

$$y_h^n(x_1, x_2) := u^n(x_1, hx_2).$$

Clearly, $y_h^n \in W^{2,2}(\Omega;\mathbb{R}^3)$. As $h \to 0$ we have $y_h^n \to u^n(\cdot, 0)$ in $W^{2,2}(\Omega;\mathbb{R}^3)$, and since $u^n(x_1,0) = y^n(x_1)$ by Theorem 2.2.15–(a), we deduce $y_h^n \to y^n$ in $W^{2,2}(\Omega;\mathbb{R}^3)$ as $h \to 0$. By Theorem 2.2.15–(b), we have that

$$\nabla_h y_h^n(x_1,x_2) = \nabla u^n(x_1, hx_2) \to \nabla u^n(x_1, 0) = \begin{pmatrix} d_1^n & d_2^n \end{pmatrix} \qquad \text{in } W^{1,2}(\Omega;\mathbb{R}^{3\times 2}).$$

Thus, $y_h^n \xrightarrow{\mathcal{F}_{\mathrm{K}}} (y^n, d_2^n)$. Finally, since u^n is an isometric immersion, we get

$$(\nabla_h y_h^n)^T \nabla_h y_h^n - \mathrm{Id} = 0. \tag{5.16}$$

Then, by Theorem 2.2.15–(c) and Proposition 5.2.4, we have

$$\begin{aligned} \int_\Omega |\nabla_h^2 y_h^n|^2\, dx &= \int_\Omega |\nabla^2 u^n(x_1, hx_2)|^2\, dx \to \int_\Omega |\nabla^2 u^n(x_1, 0)|^2\, dx \\ &= \int_\Omega |\lambda^n(x_1) p^n(x_1) \otimes p^n(x_1)|^2\, dx = \int_0^L |M^n|^2\, dx \end{aligned}$$

that gives, in view of (5.16)

$$E_h^2(y_h^n) \to \int_0^L |M_n|^2\, dx. \tag{5.17}$$

Arguing as in the proof of Theorem 5.2.2, we get that $y_n \to y$ in $W^{2,2}$ and $d_2^n \to d_2$ in $W^{1,2}$. Thus, owing to (5.15) and (5.17), by a diagonal argument, we can extract a sequence y_h such that $y_h \xrightarrow{\mathcal{F}_{\mathrm{K}}} (y, d_2)$ and

$$E_h^2(y_h) \to I^{\mathrm{S}}(d_1, d_2),$$

concluding the recovery sequence construction. For a general pair $(y, d_2) \in \mathcal{F}_{\mathrm{K}}$ we conclude by density, arguing as in proof of Theorem 5.2.2. □

Remark 5.2.8. Owing to Lemma 5.2.6, and arguing as in the proof of the liminf inequality, one can show that the lower semicontinuous envelope of Ψ, defined as in (5.10), with respect to the L^2 weak topology is

$$\Psi^*(M) := \int_0^L |M|^2\,dx + 2\int_0^L |\det(M)|^2.$$

Before moving to the Von Kármán's regime, let us give a few hints of the main difficulties we encounter when $\delta_h \sim h^2$. Arguing as in Lemma 5.2.3, one can deduce that $\det(\nabla_h^2 y_h) \overset{*}{\rightharpoonup} \mu$, for some $\mu \in \mathcal{M}_b(\Omega)$. By (5.9), it follows that $-2\mu = \partial_{22} g$ in the distributional sense, where g is the L^2 weak-limit of $h^{-2}G^h_{11}$. At the moment we do not have a full characterization of the measures μ that can be obtained in this process. However, preliminary calculations suggest that μ should belong to the space $\mathcal{M}_b(0,L;L^2(-1/2,1/2))$.

Furthermore, arguing as in the proof of Theorem 5.2.2, one can get a lower bound where the term $\|\det(\mathbb{B})\|_{L^1}$ is replaced by $\|\det(\mathbb{B}) - \mu\|_{\mathcal{M}_b}$. The constraint $-2\mu = \partial_{22} g$ seems to suggest that also the term $\|g\|^2_{L^2}$ should be kept at the limit. We believe, however, that this lower bound is too loose.

Constructions of deformations at the optimal scaling show that at least two different oscillatory behaviours can take place, competing with each other. The first is an oscillation of the Frenet–Serrin frame at a rate that is slower than h. The second is an oscillation at scale h that seems to preserve some structure of the Hessian at the limit. The main difficulty is how to detect these oscillations for an arbitrary sequence in the lower bound.

5.3 The Von Kármán's regime

We start the section by proving a compactness results that holds for every $\alpha > 2$, then we move to the proofs of Γ-convergence, dividing our results between the three regimes: $\alpha \in (2,4)$, $\alpha = 4$, and $\alpha \in (4,\infty)$.

5.3.1 Compactness

Proposition 5.3.1. *Let $\alpha > 2$ and let $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations such that $E^\alpha_h(y_h) \leq C$. Then, there is a sequence of rotations $(\bar{R}_h) \subset SO(3)$ and a sequence of constants vectors $(c_h) \subset \mathbb{R}^3$ such that, setting $\tilde{y}_h := \bar{R}_h^T y_h + c_h$, we have the following convergences (up to a subsequence):*

(i) $u_{h,1} := \min\{\delta_h^{-\gamma}, \delta_h^{2-2\gamma}\}(\tilde{y}_{h,1} - x_1) \rightharpoonup u_1$ *in* $W^{1,2}(\Omega)$,

(ii) $u_{h,2} := h\min\{\delta_h^{-\gamma}, \delta_h^{2-2\gamma}\}(\tilde{y}_{h,2} - hx_2) \rightharpoonup u_2$ *in* $W^{1,2}(\Omega)$, *for some* $u_2 \in W^{2,2}(0,L)$,

(iii) $u_{h,3} := \delta_h^{1-\gamma}\tilde{y}_{h,3} \rightharpoonup u_3$ *in* $W^{2,2}(\Omega)$, *with* $u_3 \in W^{2,2}(0,L)$,

(iv) $A_h := \delta_h^{1-\gamma}(\nabla_h \tilde{y}_h - \mathrm{Id}_{3\times 2}) \rightharpoonup A$ *in* $W^{1,2}(\Omega;\mathbb{R}^{3\times 2})$, *with* $A \in W^{1,2}(0,L;\mathbb{R}^{3\times 2})$.

Moreover, A has the form

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 & \theta \end{pmatrix},$$

with $\theta \in W^{1,2}(0, L)$. Lastly,

(a) *if $\alpha \geq 4$ we have $u_1 = \xi - x_2 \partial_1 u_2$ for some $\xi \in W^{1,2}(0, L)$,*

(b) *if $2 < \alpha < 4$ we have $u_2 = 0$ and u_1, u_3 satisfy (5.3).*

Proof. Define

$$M_h := \frac{1}{|\Omega|} \int_\Omega \nabla_h y_h \, dx \in \mathbb{R}^{3\times 2}.$$

By the Poincaré–Wirtinger inequality, we have that

$$\|\nabla_h y_h - M_h\|_{L^2} \leq C\|\nabla(\nabla_h y_h)\|_{L^2} \leq C\|\nabla_h^2 y_h\|_{L^2} \leq C\delta_h^{\gamma-1}. \tag{5.18}$$

By the uniform bound on the energy, $\nabla_h y_h$ is uniformly bounded in $W^{1,2}(\Omega; \mathbb{R}^{3\times 2})$. In particular, M_h is uniformly bounded. Since

$$M_h^T M_h - \mathrm{Id} = (M_h - \nabla_h y_h)^T M_h + \nabla_h y_h^T (M_h - \nabla_h y_h) + \nabla_h y_h^T \nabla_h y_h - \mathrm{Id},$$

it follows that

$$\begin{aligned} |M_h^T M_h - \mathrm{Id}\,| &= C\|M_h^T M_h - \mathrm{Id}\,\|_{L^1} \\ &\leq C\Big(\|\nabla_h y_h - M_h\|_{L^2}\|M_h\|_{L^2} \\ &\qquad + \|\nabla_h y_h\|_{L^2}\|\nabla_h y_h - M_h\|_{L^2} + \|\nabla_h y_h^T \nabla_h y_h - \mathrm{Id}\,\|_{L^2}\Big) \\ &\leq C\delta_h^{\gamma-1}. \end{aligned} \tag{5.19}$$

In particular, for $h \ll 1$ we have that $M_h^T M_h$ is positive definite, thus by polar decomposition there is a matrix $R_h \in \mathrm{O}(3,2)$ such that $M_h = R_h\sqrt{M_h^T M_h}$. Define

$$\bar{R}_h := \begin{pmatrix} R_h e_1 & R_h e_2 & R_h e_1 \wedge R_h e_2 \end{pmatrix}$$

Let

$$c_h := -\frac{1}{|\Omega|} \int_\Omega (\bar{R}_h^T y_h - x_1 e_1 - h x_2 e_2) \, dx,$$

and set $\tilde{y}_h := \bar{R}_h^T y_h + c_h$. Observe that $E_h^\alpha(\tilde{y}_h) = E_h^\alpha(y_h)$. We start by showing convergence (iv). Note that $\sqrt{M_h^T M_h} + \mathrm{Id}$ is invertible and thus, by (5.19),

$$\left|\sqrt{M_h^T M_h} - \mathrm{Id}\right| \leq \left|\left(\sqrt{M_h^T M_h} + \mathrm{Id}\right)^{-1}\right| |M_h^T M_h - \mathrm{Id}\,| \leq C\delta_h^{\gamma-1}. \tag{5.20}$$

Then, by definition of $\bar{R}_h$, we have

$$\bar{R}_h^T M_h = \begin{pmatrix} \sqrt{M_h^T M_h} \\ 0 \end{pmatrix}, \tag{5.21}$$

so that, by (5.18) and (5.20)

$$\begin{aligned}\|A_h\|_{L^2} &\leq \delta_h^{1-\gamma}\|\nabla_h\tilde{y}_h - \mathrm{Id}_{3\times 2}\|_{L^2}\\ &\leq \delta_h^{1-\gamma}\left(\|\tilde{R}_h^T\nabla_h y_h - \tilde{R}_h^T M_h\|_{L^2} + \|\tilde{R}_h^T M_h - \mathrm{Id}_{3\times 2}\|_{L^2}\right) \leq C.\end{aligned}$$

Moreover,

$$\|\nabla A_h\|_{L^2} = \delta_h^{1-\gamma}\|\nabla(\nabla_h\tilde{y}_h)\|_{L^2} \leq \delta_h^{1-\gamma}\|\nabla_h^2\tilde{y}_h\|_{L^2} \leq C,$$

and this concludes the proof of convergence (iv). Since

$$\|\partial_2 A_h\|_{L^2} = \delta_h^{1-\gamma}\|\partial_2(\nabla_h\tilde{y}_h)\|_{L^2} = h\delta_h^{1-\gamma}\|h^{-1}\partial_2(\nabla_h\tilde{y}_h)\|_{L^2} \leq Ch \to 0,$$

we have that A is independent of x_2.

For convergence (iii), it is sufficient to note that $u_{h,3}$ has zero mean and that

$$\begin{aligned}&\|\nabla u_{h,3}\|_{L^2} = \delta^{1-\gamma}\|\nabla\tilde{y}_{h,3}\|_{L^2} \leq \delta^{1-\gamma}\|\nabla_h\tilde{y}_{h,3}\|_{L^2} \leq \delta^{1-\gamma}\|\nabla\tilde{y}_h - \mathrm{Id}_{3\times 2}\|_{L^2} \leq C,\\ &\|\nabla^2 u_{h,3}\|_{L^2} = \delta_h^{1-\gamma}\|\nabla\nabla_h\tilde{y}_{h,3}\|_{L^2} \leq \delta_h^{1-\gamma}\|\nabla_h^2\tilde{y}_h\|_{L^2} \leq C.\end{aligned}$$

As before, to prove that u_3 is independent of x_2 note that

$$\|\partial_2 u_{h,3}\|_{L^2} = h\delta_h^{1-\gamma}\|h^{-1}\partial_2\tilde{y}_h\|_{L^2} \leq h\delta_h^{1-\gamma}\|\nabla_h\tilde{y}_h - \mathrm{Id}_{3\times 2}\|_{L^2} \to 0.$$

Moreover, since

$$\partial_1 u_{h,3} = \delta_h^{1-\gamma}\partial_1\tilde{y}_{h,3} = (A_h)_{31},$$

we have $A_{31} = \partial_1 u_3$.

We move now to the proof of (i)–(ii). Observe that

$$\begin{aligned}(\nabla_h\tilde{y}_h - \mathrm{Id}_{3\times 2})^T(\nabla_h\tilde{y}_h - \mathrm{Id}_{3\times 2}) &= \nabla_h\tilde{y}_h^T\nabla_h\tilde{y}_h - \mathrm{Id}\\ &- 2\,\mathrm{sym}((\nabla_h\tilde{y}_h)' - \mathrm{Id}).\end{aligned} \tag{5.22}$$

By (iv) and a standard Sobolev's Embedding argument, we have

$$\|\nabla_h\tilde{y}_h - \mathrm{Id}_{3\times 2}\|_{L^p} \leq C\delta_h^{\gamma-1} \qquad \forall\, p < \infty. \tag{5.23}$$

Thus, by (5.22) and Hölder's inequality we have that

$$\|\,\mathrm{sym}((\nabla_h\tilde{y}_h)' - \mathrm{Id})\|_{L^2} \leq C\max\{\delta_h^{2\gamma-2}, \delta_h^{\gamma}\}. \tag{5.24}$$

By some simple computations, we have

$$\begin{pmatrix}\partial_1 u_{h,1} & h^{-1}\partial_2 u_{h,1}\\ h^{-1}\partial_1 u_{h,2} & h^{-2}\partial_2 u_{h,2}\end{pmatrix} = \min\{\delta_h^{2-2\gamma}, \delta_h^{-\gamma}\}((\nabla_h\tilde{y}_h)' - \mathrm{Id}). \tag{5.25}$$

Taking the symmetric part on both sides of (5.25), by (5.24) we deduce that, for $h \ll 1$

$$\left\|\mathrm{sym}\begin{pmatrix}\partial_1 u_{h,1} & \partial_2 u_{h,1}\\ \partial_1 u_{h,2} & \partial_2 u_{h,2}\end{pmatrix}\right\|_{L^2} \leq \left\|\mathrm{sym}\begin{pmatrix}\partial_1 u_{h,1} & h^{-1}\partial_2 u_{h,1}\\ h^{-1}\partial_1 u_{h,2} & h^{-2}\partial_2 u_{h,2}\end{pmatrix}\right\|_{L^2} \leq C \tag{5.26}$$

By (5.25) we also deduce

$$A_h' = \max\{\delta_h^{\gamma-1}, \delta_h\} \begin{pmatrix} \partial_1 u_{h,1} & h^{-1}\partial_2 u_{h,1} \\ h^{-1}\partial_1 u_{h,2} & h^{-2}\partial_2 u_{h,2} \end{pmatrix}. \tag{5.27}$$

Note that, by (5.21), we have that

$$\int_\Omega (\nabla_h \tilde{y}_h)' \, dx = \int_\Omega (\bar{R}_h^T \nabla_h y_h)' \, dx = (\bar{R}_h^T M_h)' \in R^{2\times 2}_{\mathrm{sym}}.$$

In particular, $\operatorname{skew}(A_h')$ has zero mean, and, by (5.27), so has

$$\operatorname{skew} \begin{pmatrix} \partial_1 u_{h,1} & \partial_2 u_{h,1} \\ \partial_1 u_{h,2} & \partial_2 u_{h,2} \end{pmatrix}.$$

Then, since both $u_{h,1}$ and $u_{h,2}$ have zero mean, convergences (i)–(ii) follow from (5.26) and an application of Korn's inequality. By (5.26), we also deduce that

$$\partial_2 u_2 = \partial_1 u_2 + \partial_2 u_1 = 0.$$

This implies that u_2 does not depend on x_2 and that there exists $\xi \in W^{1,2}(0, L)$ such that $u_1 = \xi - x_2 \partial_1 u_2$. In particular, $x_2 \partial_1 u_2 \in W^{1,2}(\Omega)$, thus $u_2 \in W^{2,2}(0, L)$. Taking the symmetric part on both sides of (5.27) and passing to the limit, we deduce from (5.26) that $\operatorname{sym}(A') = 0$. We show now that $A_{21} = -A_{12} = 0$. We have

$$\begin{aligned} \partial_1 (A_h)_{12} &= \partial_1(\delta_h^{1-\gamma} h^{-1} \partial_2 \tilde{y}_{h,1}) = h^{-1}\delta_h^{1-\gamma} \partial_{12} \tilde{y}_{h,1} = h^{-1}\delta_h^{1-\gamma} \partial_{12}\tilde{y}_h \cdot e_1 \\ &= h^{-1}\delta_h^{1-\gamma}\partial_{12}\tilde{y}_h \cdot \partial_1 \tilde{y}_h + h^{-1}\delta_h^{1-\gamma}\partial_{12}\tilde{y}_h \cdot (e_1 - \partial_1 \tilde{y}_h) \\ &= \frac{1}{2} h^{-1}\delta_h^{1-\gamma}\partial_2(\partial_1\tilde{y}_h \cdot \partial_1 \tilde{y}_h - 1) + h^{-1}\delta_h^{1-\gamma}\partial_{12}\tilde{y}_h(e_1 - \partial_1\tilde{y}_h). \end{aligned} \tag{5.28}$$

Note that, since $h^{-1}\delta_h \to 0$,

$$h^{-1}\delta_h^{1-\gamma}\partial_2(\partial_1\tilde{y}_h \cdot \partial_1\tilde{y}_h - 1) = h^{-1}\delta_h \partial_2[\delta_h^{-\gamma}(\partial_1\tilde{y}_h \cdot \partial_1 \tilde{y}_h - 1)] \to 0$$

in $(W^{1,2}(\Omega))^*$. Moreover, since $\delta_h^{1-\gamma} h^{-1}\partial_{12}\tilde{y}_h$ is uniformly bounded in L^2, we have by (5.23)

$$h^{-1}\delta_h^{1-\gamma}\partial_{12}\tilde{y}_h(e_1 - \partial_1 \tilde{y}_h) \to 0 \qquad \text{in } L^q(\Omega),$$

for every $q \in [1, 2)$. Passing to the limit in (5.28), we deduce that $\partial_1 A_{12} = 0$, that is $A_{12} = c$ for some $c \in \mathbb{R}$. Thus, since we already observed that $\operatorname{skew}(A_h')$ has zero mean, so has A, from which it follows that $A_{12} = 0$. We are left to prove (5.3). Thus, suppose that $2 < \alpha < 4$. Then, dividing (5.22) by $\delta_h^{2\gamma-2}$ and passing to the limit, we deduce

$$\delta_h^{2-2\gamma} \operatorname{sym}((\nabla_h y_h)' - \mathrm{Id}) \rightharpoonup -\frac{1}{2} A^T A.$$

Looking at the top-left component, we get

$$\partial_1 u_1 = -\frac{1}{2}(A^T A)_{11} = -\frac{1}{2} A_{31}^2 = -\frac{1}{2}(\partial_1 u_3)^2,$$

proving (5.3). Recalling that $u_1 = \xi - x_2 \partial_1 u_2$, and that u_3 is independent of x_2, we immediately deduce that $\partial_1 u_2 = 0$. Since u_2 has zero mean, it follows that $u_2 = 0$. Then, by (5.3), $u_1 \in W^{2,2}(\Omega)$. □

5.3.2 Γ-convergence for $\alpha > 4$

In order to make the exposition more clear, we split the Γ-convergence result in two parts. Firstly, we prove the Γ-liminf inequality, then we move to the recovery sequence construction.

Theorem 5.3.2 (lim inf inequality)**.** *Let $\alpha > 4$. For any sequence of deformations $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that $E_h^\alpha(y_h) \leq C$ we have that*

$$\liminf_{h\to 0^+} E_h^\alpha(y_h) \geq I^{\mathrm{LVK}}(u_1,u_3,\theta),$$

where u_1, u_3 and θ are the ones given by Proposition 5.3.1.

Proof. We borrow the notation from Proposition 5.3.1. Firstly, note that $E_h^\alpha(y_h) = E_h^\alpha(\tilde{y}_h)$. Moreover, by the convergence of A_h, we deduce

$$\|\nabla_h \tilde{y}_h - \mathrm{Id}_{3\times 2}\|_{L^2} \leq C\delta_h^{\gamma-1}.$$

Thus, since $\gamma > 2$, we have

$$\frac{1}{\delta_h^\gamma}(\nabla_h\tilde{y}_h - \mathrm{Id}_{3\times 2})^T(\nabla_h\tilde{y}_h - \mathrm{Id}_{3\times 2}) \to 0$$

strongly in $L^2(S;\mathbb{R}^{2\times 2})$. Hence, by (5.22) it follows that

$$\frac{1}{\delta_h^\gamma}(\nabla_h\tilde{y}_h^T\nabla_h\tilde{y}_h - \mathrm{Id})_{11} \rightharpoonup 2\partial_1 u_1 \quad \text{in } L^2(\Omega).$$

Then, by lower semicontinuity

$$\begin{aligned}
\liminf_{h\to 0^+} E_h^\alpha(y_h) &= \liminf_{h\to 0^+} E_h^\alpha(\tilde{y}_h) \\
&\geq \liminf_{h\to 0^+}\left[\frac{1}{\delta_h^\alpha}\int_\Omega |(\nabla_h\tilde{y}_h^T\nabla_h\tilde{y}_h - \mathrm{Id})_{11}|^2\,dx\right. \\
&\qquad\left. + \frac{1}{\delta_h^{\alpha-2}}\int_\Omega |\partial_{11}\tilde{y}_{h,3}|^2\,dx + \frac{2}{\delta_h^{\alpha-2}}\int_\Omega |h^{-1}\partial_{12}\tilde{y}_{h,3}|^2\,dx\right] \\
&\geq 4\int_\Omega |\partial_1 u_1|^2\,dx + \int_0^L \left(|\partial_{11}u_3|^2 + 2|\partial_1\theta|^2\right)dx = I^{\mathrm{LVK}}(u_1,u_3,\theta).
\end{aligned}$$

This concludes the proof. □

Theorem 5.3.3 (Recovery sequence)**.** *Let $\alpha > 4$. Let $\theta \in W^{1,2}(0,L)$, $u_2, u_3 \in W^{2,2}(0,L)$ and $u_1 \in W^{1,2}(\Omega)$ such that $u_1 = \xi - x_2\partial_1 u_2$ for some $\xi \in W^{1,2}(0,L)$. Then, there exists a sequence of deformations $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that*

(i) *$u_{h,1} := \delta_h^{-\gamma}(y_{h,1} - x_1) \to u_1 = \xi - x_2\partial_1 u_2$ in $W^{1,2}(\Omega)$,*

(ii) *$u_{h,2} := h\delta_h^{-\gamma}(y_{h,2} - hx_2) \to u_2$ in $W^{2,2}(\Omega)$, with $u_2 \in W^{2,2}(0,L)$,*

(iii) *$u_{h,3} := \delta_h^{1-\gamma}y_{h,3} \to u_3$ in $W^{2,2}(\Omega)$ with $u_3 \in W^{2,2}(0,L)$,*

(iv) $A_h := \delta_h^{1-\gamma}(\nabla_h y_h - \mathrm{Id}_{3\times 2}) \to A$ *in* $W^{1,2}(\Omega;\mathbb{R}^{3\times 2})$ *with*

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 & \theta \end{pmatrix},$$

(v) $E_h^\alpha(y_h) \to I^{\mathrm{LVK}}(u_1, u_3, \theta)$.

Proof. Suppose that ξ, u_2, u_3 and θ are in $C^\infty([0,L])$. Define

$$y_h := \begin{pmatrix} x_1 \\ hx_2 \\ 0 \end{pmatrix} + \delta_h^{\gamma-1}\begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} + \frac{\delta_h^\gamma}{h}\begin{pmatrix} 0 \\ u_2 \\ 0 \end{pmatrix} + hx_2\delta_h^{\gamma-1}\begin{pmatrix} 0 \\ 0 \\ \theta \end{pmatrix} + \delta_h^\gamma\begin{pmatrix} \xi - x_2\partial_1 u_2 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly $y_h \in W^{2,2}(S)$. The convergences of $u_{h,1}, u_{h,2}$ and $u_{h,3}$ follow trivially since $h^{-1}\delta_h \to 0$. We have

$$\nabla_h y_h = \mathrm{Id}_{3\times 2} + \delta_h^{\gamma-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 & \theta \end{pmatrix} + \frac{\delta_h^\gamma}{h}\begin{pmatrix} 0 & -\partial_1 u_2 \\ \partial_1 u_2 & 0 \\ 0 & 0 \end{pmatrix} + hx_2\delta_h^{\gamma-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1\theta & 0 \end{pmatrix}$$
$$+ \delta_h^\gamma\begin{pmatrix} \partial_1\xi - x_2\partial_{11}u_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $h^{-1}\delta_h \to 0$, the convergence of A_h follows. Moreover, we have

$$\nabla_h y_h^T \nabla_h y_h - \mathrm{Id} = \delta_h^\gamma \begin{pmatrix} 2(\partial_1\xi - x_2\partial_{11}u_2) & 0 \\ 0 & 0 \end{pmatrix} + o(\delta_h^\gamma).$$

Differentiating the rescaled gradient, we have that

$$\partial_1 \nabla_h y_h = \delta_h^{\gamma-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_{11}u_3 & \partial_1\theta \end{pmatrix} + o(\delta_h^{\gamma-1}),$$
$$\frac{1}{h}\partial_2 \nabla_h y_h = \delta_h^{\gamma-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1\theta & 0 \end{pmatrix} + o(\delta_h^{\gamma-1}).$$

This easily implies that

$$E_h^\alpha(y_h) \to 4\int_\Omega |\partial_1 u_1|^2\,dx + \int_0^L |\partial_{11}u_3|^2\,dx + 2\int_0^L |\partial_1\theta|^2\,dx = I^{\mathrm{LVK}}(u_1,u_3,\theta).$$

By a standard density argument the proof is concluded. □

5.3.3 Γ-convergence for $\alpha = 4$

Theorem 5.3.4 (lim inf inequality). *Let* $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ *be such that* $E_h^4(y_h) \le C$. *Then*

$$\liminf_{h\to 0^+} E_h^4(y_h) \ge I^{\mathrm{VK}}(u_1,u_3,\theta).$$

where u_1, u_3, *and* θ *are the one given by Proposition 5.3.1.*

Proof. Let $\tilde{y}_h$ be the deformation given by Proposition 5.3.1. Firstly, note that $E_h^4(y_h) = E_h^4(\tilde{y}_h)$. Moreover, by the weak convergence of A_h in $W^{1,2}(\Omega;\mathbb{R}^{3\times 2})$ we have

$$\delta_h^{-1}(\nabla_h \tilde{y}_h - \mathrm{Id}_{3\times 2}) \to A \quad \text{in } L^p(\Omega;\mathbb{R}^{3\times 2})$$

for every $p < \infty$. Then

$$\frac{1}{\delta_h^2}(\nabla_h \tilde{y}_h - \mathrm{Id}_{3\times 2})^T(\nabla_h \tilde{y}_h - \mathrm{Id}_{3\times 2}) \to A^T A$$

strongly in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Thus, by (5.22) it follows that

$$\frac{1}{\delta_h^2}(\nabla_h \tilde{y}_h^T \nabla_h \tilde{y}_h - \mathrm{Id})_{11} \rightharpoonup (A^T A)_{11} + 2\partial_1 u_1 = (\partial_1 u_3)^2 + 2\partial_1 u_1 \quad \text{in } L^2(\Omega).$$

Then,

$$\begin{aligned}
\liminf_{h\to 0^+} E_h^4(y_h) &= \liminf_{h\to 0^+} E_h^4(\tilde{y}_h) \\
&\geq \liminf_{h\to 0^+} \Bigg[\frac{1}{\delta_h^4}\int_\Omega |(\nabla_h \tilde{y}_h^T \nabla_h \tilde{y}_h - \mathrm{Id})_{11}|^2\,dx \\
&\qquad\qquad + \frac{1}{\delta_h^2}\int_\Omega |\partial_{11}\tilde{y}_{h,3}|^2\,dx + \frac{2}{\delta_h^2}\int_\Omega |h^{-1}\partial_{12}\tilde{y}_{h,3}|^2\,dx \Bigg] \\
&\geq \int_\Omega |(\partial_1 u_3)^2 + 2\partial_1 u_1|^2\,dx + \int_0^L \left(|\partial_{11}u_3|^2 + 2|\partial_1\theta|^2\right)dx \\
&= I^{\mathrm{VK}}(u_1,u_3,\theta).
\end{aligned}$$

□

Theorem 5.3.5 (Recovery sequence). *Let $\theta \in W^{1,2}(0,L)$, $u_2, u_3 \in W^{2,2}(0,L)$ and $u_1 \in W^{1,2}(\Omega)$ such that $u_1 = \xi - x_2\partial_1 u_2$ for some $\xi \in W^{1,2}(0,L)$. Then, there exists a sequence of deformations $(y_h)_h \subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that*

(i) $u_{h,1} := \delta_h^{-2}(y_{h,1} - x_1) \to u_1 = \xi - x_2\partial_1 u_2$ *in* $W^{1,2}(\Omega)$,

(ii) $u_{h,2} := h\delta_h^{-2}(y_{h,2} - hx_2) \to u_2$ *in* $W^{2,2}(\Omega)$, *with* $u_2 \in W^{2,2}(0,L)$,

(iii) $u_{h,3} := \delta_h^{-1} y_{h,3} \to u_3$ *in* $W^{2,2}(\Omega)$ *with* $u_3 \in W^{2,2}(0,L)$,

(iv) $A_h := \delta_h^{-1}(\nabla_h y_h - \mathrm{Id}_{3\times 2}) \to A$ *in* $W^{1,2}(\Omega;\mathbb{R}^{3\times 2})$ *with*

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 & \theta \end{pmatrix},$$

(v) $E_h^4(y_h) \to I^{\mathrm{VK}}(u_1,u_3,\theta)$.

Proof. Suppose that ξ, u_2, u_3 and θ are in $C^\infty([0,L])$. Define

$$y_h := \begin{pmatrix} x_1 \\ hx_2 \\ 0 \end{pmatrix} + \delta_h \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix} + \frac{\delta_h^2}{h} \begin{pmatrix} 0 \\ u_2 \\ 0 \end{pmatrix} + hx_2\delta_h \begin{pmatrix} 0 \\ 0 \\ \theta \end{pmatrix} + \delta_h^2 \begin{pmatrix} \xi - x_2\partial_1 u_2 \\ 0 \\ 0 \end{pmatrix}$$
$$- h\delta_h^2 x_2 \begin{pmatrix} \partial_1 u_3 \theta \\ \theta^2/2 \\ 0 \end{pmatrix}.$$

The convergences of $u_{h,1}, u_{h,2}$ and $u_{h,3}$ follow trivially since $h^{-1}\delta_h \to 0$. We have

$$\nabla_h y_h = \mathrm{Id}_{3\times 2} + \delta_h \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ u_3' & \theta \end{pmatrix} + \frac{\delta_h^2}{h} \begin{pmatrix} 0 & -\partial_1 u_2 \\ \partial_1 u_2 & 0 \\ 0 & 0 \end{pmatrix} + hx_2\delta_h \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1\theta & 0 \end{pmatrix}$$
$$+ \delta_h^2 \begin{pmatrix} \partial_1\xi - x_2\partial_{11}u_2 & -\partial_1 u_3\theta \\ 0 & -\theta^2/2 \\ 0 & 0 \end{pmatrix} + O(h\delta_h^2),$$

and

$$\partial_1 \nabla_h y_h = \delta_h \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_{11}u_3 & \partial_1\theta \end{pmatrix} + o(\delta_h, L^2),$$
$$\frac{1}{h}\partial_2 \nabla_h y_h = \delta_h \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1\theta & 0 \end{pmatrix} + o(\delta_h, L^2).$$

In particular, we have the desired convergence of A_h. By some simple computation, we have

$$\nabla_h y_h^T \nabla_h y_h - \mathrm{Id} = \delta_h^2 \begin{pmatrix} 2(\partial_1\xi - x_2\partial_{11}u_2) + (\partial_1 u_3)^2 & \partial_1 u_3\theta - \partial_1 u_3\theta \\ \partial_1 u_3\theta - \partial_1 u_3\theta & \theta^2 - \theta^2 \end{pmatrix} + o(\delta_h^2)$$
$$= \delta_h^2 \begin{pmatrix} 2(\partial_1\xi - x_2\partial_{11}u_2) + (\partial_1 u_3)^2 & 0 \\ 0 & 0 \end{pmatrix} + o(\delta_h^2).$$

Putting all the calculations together we conclude that $E_h^4(y_h) \to I^{\mathrm{VK}}(u_1, u_3, \theta)$. For an arbitrary quadruplet (ξ, u_2, u_3, θ) we conclude by density. □

5.3.4 Γ-convergence for $2 < \alpha < 4$

This section is further divided between the two cases $h^{-2}\delta_h^{2-\gamma} \to 0, +\infty$. We start with the simpler one, in which no constraint on the Gauss' curvature is present.

Γ-convergence for $h^{-2}\delta_h^{2-\gamma} \to +\infty$

Theorem 5.3.6 (lim inf inequality)**.** *Let $2 < \alpha < 4$. For every sequence of deformations $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that $E_h^\alpha(y_h) \le C$ it holds*

$$\liminf_{h\to 0^+} E_h^\alpha(y_h) \ge I^{\mathrm{CVK}}(u_3, \theta),$$

where u_3, and θ are the ones given by Proposition 5.3.1.

Proof. Let $\tilde{y}_h$ be the deformation given by Proposition 5.3.1. We have

$$\begin{aligned}\liminf_{h\to 0^+} E_h^\alpha(y_h) &= \liminf_{h\to 0^+} E_h^\alpha(\tilde{y}_h)\\ &\geq \liminf_{h\to 0}\left[\frac{1}{\delta_h^{\alpha-2}}\int_\Omega |\partial_{11}\tilde{y}_{h,3}|^2\,dx + \frac{2}{\delta_h^{\alpha-2}}\int_\Omega |h^{-1}\partial_{12}\tilde{y}_{h,3}|^2\,dx\right]\\ &\geq \int_0^L |\partial_{11}u_3|^2\,dx + 2\int_0^L |\partial_1\theta|^2\,dx = I^{\mathrm{CVK}}(u_3,\theta),\end{aligned}$$

that concludes the proof. □

Theorem 5.3.7 (Recovery sequence)**.** *Let $2 < \alpha < 4$ and suppose that $h^{-2}\delta_h^{2-\gamma} \to +\infty$. Let $\theta \in W^{1,2}(0,L)$ and $u_3 \in W^{2,2}(0,L;\mathbb{R}^3)$. Let $u_1 \in W^{2,2}(0,L)$ such that $2\partial_1 u_1 + (\partial_1 u_3)^2 = 0$. Then, there exists a sequence of deformations $(y_h) \subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that*

(i) $u_{h,1} := \delta_h^{2-2\gamma}(y_{h,1} - x_1) \to u_1$ *in* $W^{1,2}(\Omega)$ *with* $u_1 \in W^{2,2}(0,L)$,

(ii) $u_{h,2} := h\delta_h^{2-2\gamma}(y_{h,2} - hx_2) \to 0$ *in* $W^{2,2}(\Omega)$,

(iii) $u_{h,3} := \delta_h^{1-\gamma} y_{h,3} \to u_3$ *in* $W^{2,2}(\Omega)$ *with* $u_3 \in W^{2,2}(0,L)$,

(iv) $A_h := \delta_h^{1-\gamma}(\nabla_h y_h - \mathrm{Id}_{3\times 2}) \to A$ *in* $W^{1,2}(\Omega;\mathbb{R}^{3\times 2})$ *with*

$$A = \begin{pmatrix} 0 & 0\\ 0 & 0\\ \partial_1 u_3 & \theta\end{pmatrix},$$

(v) $E_h^\alpha(y_h) \to I^{\mathrm{CVK}}(u_3,\theta)$.

Proof. Suppose that $u_3, \theta \in C^\infty([0,L])$, so that $u_1 \in C^\infty([0,L])$. Let

$$\tilde{A} := \begin{pmatrix} 0 & 0 & -\partial_1 u_3\\ 0 & 0 & -\theta\\ \partial_1 u_3 & \theta & 0\end{pmatrix}.$$

We denote by $R_h \in C^\infty([0,L])$ the unique global solution of the Cauchy's problem

$$\begin{cases}\partial_1 X = \delta_h^{\gamma-1} X\partial_1\tilde{A} & \text{in } [0,L],\\ X(0) = \exp(\delta_h^{\gamma-1}\tilde{A}(0)).\end{cases}$$

Note that $R^h(0) \in SO(3)$ and that

$$\partial_1(R_h R_h^T) = \partial_1 R_h R_h^T + R_h(\partial_1 R_h)^T = \delta_h^{\gamma-1}(R_h\partial_1\tilde{A}R_h^T + R_h(\partial_1\tilde{A})^T R_h^T) = 0,$$

that is $R_h \in SO(3)$ everywhere in $[0,L]$. We define

$$y_h := \int_0^{x_1} R_h(t)e_1\,dt + hx_2R_he_2 + \delta_h^{\gamma-1}\begin{pmatrix}\delta_h^{\gamma-1}u_1(0)\\ 0\\ u_3(0)\end{pmatrix}.$$

We have

$$
\begin{aligned}
\nabla_h y_h &= \begin{pmatrix} R_h e_1 & R_h e_2 \end{pmatrix} + h x_2 \partial_1 R_h \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} R_h e_1 & R_h e_2 \end{pmatrix} + h \delta_h^{\gamma-1} x_2 R_h \partial_1 \tilde{A} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} R_h e_1 & R_h e_2 \end{pmatrix} + h \delta_h^{\gamma-1} x_2 R_h \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 \theta & 0 \end{pmatrix}.
\end{aligned}
$$

In particular, we deduce that

$$
R_h^T \nabla_h y_h = \mathrm{Id}_{3\times 2} + h x_2 \delta_h^{\gamma-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \theta' & 0 \end{pmatrix}.
$$

Then

$$
\begin{aligned}
\nabla_h y_h^T \nabla_h y_h - \mathrm{Id} &= (R_h^T \nabla_h y_h)^T (R_h^T \nabla_h y_h) - \mathrm{Id} \\
&= h^2 \delta_h^{2\gamma-2} x_2^2 \begin{pmatrix} (\partial_1 \theta)^2 & 0 \\ 0 & 0 \end{pmatrix} + O(h^2 \delta_h^{2\gamma-2}).
\end{aligned}
$$

By the assumption $h^{-2}\delta_h^{2-\gamma} \to +\infty$ we get that

$$
\frac{1}{\delta_h^\alpha} \int_\Omega |\nabla_h^T y_h \nabla_h y_h - \mathrm{Id}\,|^2 \, dx \to 0.
$$

We move now to the computation of the rescaled Hessian. We have

$$
\begin{aligned}
\partial_1(\nabla_h y_h) &= \begin{pmatrix} \partial_1 R_h e_1 & \partial_1 R_h e_2 \end{pmatrix} + \delta_h^{\gamma-1} R_h \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_{11} u_3 & \partial_1 \theta \end{pmatrix} + o(\delta_h^{\gamma-1}), \\
\frac{1}{h}\partial_2(\nabla_h y_h) &= \delta_h^{\gamma-1} R_h \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 \theta & 0 \end{pmatrix} + o(\delta_h^{\gamma-1}).
\end{aligned}
$$

Since rotations do not change the Frobenius' norm, this proves that $E_h^\alpha(y_h) \to I^{\mathrm{CVK}}(u_3, \theta)$. We are left to prove the convergences of $u_{h,1}, u_{h,2}, u_{h,3}$ and A_h. Following [FMP13, Theorem 3.12] we show that R_h has the following structure:

$$
R_h = \mathrm{Id} + \delta_h^{\gamma-1} \tilde{A} + \delta_h^{2\gamma-2} \int_0^{x_1} \tilde{A}(s) \partial_1 \tilde{A}(s) \, ds + \frac{1}{2} \delta_h^{2\gamma-2} \tilde{A}^2(0) + O(\delta^{3\gamma-3}). \quad (5.29)
$$

Indeed, define

$$
Q_h(x_1) := \mathrm{Id} + \delta_h^{\gamma-1} \tilde{A}(x_1) + \delta_h^{2\gamma-2} \int_0^{x_1} \tilde{A}(s) \partial_1 \tilde{A}(s) \, ds + \frac{1}{2} \delta_h^{2\gamma-2} \tilde{A}^2(0).
$$

Observe that Q_h solves the following ODE

$$\partial_1 Q_h = \delta_h^{\gamma-1} Q_h \partial_1 \tilde{A} - \delta_h^{3\gamma-3} \left(\int_0^{x_1} \tilde{A}(s) \partial_1 \tilde{A}(s)\, ds + \frac{1}{2} \tilde{A}^2(0) \right) \partial_1 \tilde{A}$$

with initial datum

$$Q_h(0) = \exp(\delta_h^{\gamma-1} \tilde{A}(0)) + O(\delta_h^{3\gamma-3}).$$

Then, by Gronwall's Lemma 2.2.3

$$|Q_h - R^h| = O(\delta_h^{3\gamma-3}).$$

Recalling that $2\partial_1 u_1 = -(\partial_1 u_3)^2$, by (5.29) we have

$$\begin{aligned}(R_h)_{11} &= 1 - \delta_h^{2\gamma-2} \int_0^{x_1} \partial_1 u_3 \partial_{11} u_3\, ds - \frac{1}{2} \delta_h^{2\gamma-2} (\partial_1 u_3)^2(0) + O(\delta_h^{3\gamma-3}) \\ &= 1 + \delta_h^{2\gamma-2} u_1'(x_1) + O(\delta_h^{3\gamma-3}),\end{aligned} \tag{5.30}$$

$$(R_h)_{12} = O(\delta_h^{2\gamma-2}). \tag{5.31}$$

Then, by (5.30)–(5.31) we have

$$\begin{aligned}u_{h,1} &= \frac{1}{\delta_h^{2\gamma-2}} (y_{h,1} - x_1) = \frac{1}{\delta_h^{2\gamma-2}} \left[\int_0^{x_1} (R_h)_{11}\, ds + h x_2 (R_h)_{12} - x_1 \right] + u_1(0) \\ &= u_1 - u_1(0) + u_1(0) + O(h, \delta_h^{\gamma-1}) \to u_1\end{aligned}$$

in $W^{1,2}(S)$. To deduce the rest of the convergences we argue similarly. We have

$$\begin{aligned}(R_h)_{21} &= O(\delta_h^{2\gamma-2}), \\ (R_h)_{22} &= 1 + O(\delta_h^{2\gamma-2}).\end{aligned}$$

Thus,

$$\begin{aligned}u_{h,2} &= \frac{h}{\delta_h^{2\gamma-2}} (y_{h,2} - h x_2) = \frac{h}{\delta_h^{2\gamma-2}} \left[\int_0^{x_1} (R_h)_{21}(s)\, ds + h x_2 (R_h)_{22} - h x_2 \right] \\ &= O(h) \to 0\end{aligned}$$

in $W^{2,2}(S)$. Lastly,

$$\begin{aligned}(R_h)_{31} &= \delta_h^{\gamma-1} \partial_1 u_3 + O(\delta_h^{2\gamma-2}), \\ (R_h)_{32} &= O(\delta_h^{\gamma-1}),\end{aligned}$$

from which we deduce

$$\begin{aligned}u_{h,3} &= \frac{1}{\delta_h^{\gamma-1}} y_{h,3} = \frac{1}{\delta_h^{\gamma-1}} \left[\int_0^{x_1} (R_h)_{31}(s)\, ds + h x_2 (R_h)_{32} \right] + u_3(0) \\ &= u_3 - u_3(0) + u_3(0) + O(h) \to u_3\end{aligned}$$

in $W^{2,2}(S)$. To conclude observe that

$$\nabla_h y_h - \mathrm{Id}_{3\times 2} = \delta_h^{\gamma-1} A + o(\delta_h^{\gamma-1}),$$

so that the convergence of A_h follows. To extend these results to non-smooth u_3 and θ we just exploit the density of smooth functions in $W^{2,2}(0,L)$ and $W^{1,2}(0,L)$ paired with the continuity of I^{CVK} with respect to the strong topology of those spaces. $\square$

Γ-convergence for $h^{-2}\delta_h^{2-\gamma} \to 0$

We start by proving a Lemma similar in spirit to Lemma 5.2.3, concerned with the convergence of a suitable rescaling of the Gauss' curvature.

Lemma 5.3.8. *Let $2 < \alpha < 4$ and suppose that $h^{-2}\delta_h^{2-\gamma} \to 0$. Let $(y_h) \subset W^{2,2}(S;\mathbb{R}^3)$ be a sequence of deformations such that $E_h^\alpha(y_h) \le C$. Then*

$$\delta_h^{2-2\gamma} \det(\nabla_h^2 y_h) \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathcal{M}_b(S).$$

Proof. Recall that, defining $G^h := \nabla_h y_h^T \nabla_h y_h - \mathrm{Id}$, equality (5.9) holds in the sense of distributions. In particular, since $\|G_h\|_{L^2} \le C\delta_h^\gamma$, we have that

$$\delta_h^{2-2\gamma}\|\det(\nabla_h^2 y_h)\|_{(W^{2,2})^*} \le C\frac{1}{h^2}\delta_h^{2-2\gamma}\|\nabla^2 G^h\|_{(W^{2,2})^*} \le Ch^{-2}\delta_h^{2-\gamma} \to 0.$$

By the boundedness of the energy, we have that $\delta_h^{2-2\gamma}\|\det(\nabla_h^2 y_h)\|_{L^1} \le C$, that gives, up to a subsequence, $\delta_h^{2-2\gamma}\det(\nabla_h^2 y_h) \overset{*}{\rightharpoonup} \mu = 0$ in $\mathcal{M}_b(S)$. $\square$

Theorem 5.3.9 (lim inf inequality). *Let $2 < \alpha < 4$ and suppose that $h^{-2}\delta_h^{2-\gamma} \to 0$.*

$$\liminf_{h\to 0^+} E_h^\alpha(y_h) \ge I^{\mathrm{VKS}}(u_3, \theta),$$

where u_3 and θ are the one given by Proposition 5.3.1.

Proof. Let $\tilde{y}_h$ be the deformation given by Proposition 5.3.1. For every $\varphi \in C_0^1(\Omega;[0,1])$ we get

$$\begin{aligned}
E_h^\alpha(\tilde{y}_h) &\ge \frac{1}{\delta_h^{\alpha-2}}\int_\Omega |\nabla_h^2 \tilde{y}_h|^2\,dx \\
&= \frac{1}{\delta_h^{\alpha-2}}\int_\Omega \varphi(|\partial_{11}\tilde{y}_{h,3}|^2 + 2|h^{-1}\partial_{12}\tilde{y}_{h,3}|^2 + |h^{-2}\partial_{22}\tilde{y}_{h,3}|^2)\,dx \\
&\quad + \frac{1}{\delta_h^{\alpha-2}}\int_\Omega (1-\varphi)(|\partial_{11}\tilde{y}_{h,3}|^2 + 2|h^{-1}\partial_{12}\tilde{y}_{h,3}|^2 + |h^{-2}\partial_{22}\tilde{y}_{h,3}|^2)\,dx \\
&= \frac{1}{\delta_h^{\alpha-2}}\int_\Omega \varphi|\partial_{11}\tilde{y}_h + h^{-2}\partial_{22}\tilde{y}_h|^2\,dx - 2\delta_h^{2-2\gamma}\int_\Omega \varphi\det(\nabla_h^2\tilde{y}_h)\,dx \\
&\quad + \frac{1}{\delta_h^{\alpha-2}}\int_\Omega (1-\varphi)(|\partial_{11}\tilde{y}_h - h^{-2}\partial_{22}\tilde{y}_h|^2 + 4|h^{-1}\partial_{12}\tilde{y}_h|^2)\,dx \\
&\quad + 2\delta_h^{2-2\gamma}\int_\Omega (1-\varphi)\det(\nabla_h^2\tilde{y}_h)\,dx.
\end{aligned} \tag{5.32}$$

Since φ takes values in $[0,1]$ all the quadratic terms are lower semicontinuous with respect to the weak convergences in L^2. Let η be the weak limit of $\delta_h^{1-\gamma}\partial_{22}\tilde{y}_{h,3}$ in $L^2(\Omega)$. Note that η exists, up to extracting a subsequence, by the boundedness of the energy. Thus, passing to the limit in (5.32), we get by Lemma 5.3.8

$$
\begin{aligned}
\liminf_{h\to 0} E_h^\alpha(y_h) &= \liminf_{h\to 0} E_h^\alpha(\tilde{y}_h) \\
&\geq \int_\Omega \varphi|\partial_{11}u_3+\eta|^2\,dx \\
&\quad + \int_\Omega (1-\varphi)\left(|\partial_{11}u_3-\eta|^2+4|\partial_{11}\theta|^2\right)\,dx \\
&= \int_0^L |\partial_{11}u_3|^2\,dx + 2\int_\Omega \varphi\partial_{11}u_3\eta\,dx + \int_\Omega |\eta|^2\,dx \\
&\quad + 4\int_0^L (1-\varphi)|\partial_{11}\theta|^2\,dx - 2\int_\Omega (1-\varphi)\partial_{11}u_3\eta\,dx \\
&= \int_0^L |\partial_{11}u_3|^2\,dx + 2\int_0^L |\partial_1\theta|^2\,dx + \int_\Omega |\eta|^2\,dx \\
&\quad + 2\int_\Omega \varphi(\partial_{11}u_3\eta-(\partial_1\theta)^2)\,dx \\
&\quad - 2\int_\Omega (1-\varphi)(\partial_{11}u_3\eta-(\partial_1\theta)^2).
\end{aligned}
\tag{5.33}
$$

Let

$$\Omega^+ := \{x\in\Omega\colon \partial_{11}u_3\eta-(\partial_1\theta)^2\geq 0\}$$

and $\Omega^- := \Omega\backslash\Omega^+$. Picking a sequence $(\varphi_n)\subset C_0^1(\Omega;[0,1])$ such that $\varphi_n\to\chi_{\Omega^+}$ almost everywhere, from (5.33) we deduce by Dominated Convergence Therorem

$$
\begin{aligned}
\liminf_{h\to 0} E_h^\alpha(y_h) \geq \int_0^L |\partial_{11}u_3|^2\,dx + 2\int_0^L |\partial_1\theta|^2\,dx + \int_\Omega |\eta|^2\,dx \\
+2\int_\Omega |\partial_{11}u_3\eta-(\partial_1\theta)^2|\,dx \geq I^{\mathrm{VKS}}(u_3,\theta).
\end{aligned}
$$

This concludes the proof. □

As for the analogous Kirchhoff's regime, in order to prove the Γ-convergence, we need to construct isometries of thin strips. The arguments are similar to the one of Theorem 5.2.2: we use a diagonal argument to relax the constraint on the rescaled Gauss' curvature. However, in this setting, the isometries we construct depend on both indices: the one linked to the relaxation and the width of the thin strip. Thus, we need a careful analysis to show that we can extract a converging diagonal subsequence.

Theorem 5.3.10 (Recovery sequence)**.** *Let $2<\alpha<4$ and suppose that $h^{-2}\delta_h^{2-\gamma}\to 0$. Let $\theta\in W^{1,2}(0,L)$ and $u_3\in W^{2,2}(0,L;\mathbb{R}^3)$. Let $u_1\in W^{2,2}(0,L)$ such that*

$$2\partial_1u_1+(\partial_1u_3)^2=0.$$

Then, there exists a sequence of deformations $(y_h)\subset W^{2,2}(\Omega;\mathbb{R}^3)$ such that

(i) $u_{h,1}:=\delta_h^{2-2\gamma}(y_{h,1}-x_1)\rightharpoonup u_1$ *in* $W^{1,2}(\Omega)$ *with* $u_1\in W^{2,2}(0,L)$*,*

(ii) $u_{h,2} := h\delta_h^{2-2\gamma}(y_{h,2} - hx_2) \rightharpoonup 0$ *in* $W^{2,2}(\Omega)$,

(iii) $u_{h,3} := \delta_h^{1-\gamma} y_{h,3} \rightharpoonup u_3$ *in* $W^{2,2}(\Omega)$ *with* $u_3 \in W^{2,2}(0,L)$,

(iv) $A_h := \delta_h^{1-\gamma}(\nabla_h y_h - \mathrm{Id}_{3\times 2}) \rightharpoonup A$ *in* $W^{1,2}(\Omega;\mathbb{R}^{3\times 2})$ *with*

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_1 u_3 & \theta \end{pmatrix},$$

(v) $E_h^\alpha(y_h) \to I^{\mathrm{VKS}}(u_3,\theta)$.

Proof. For almost every $x_1 \in (0,L)$, let $c(x_1)$ be such that

$$|c(x_1)|^2 + 2|\partial_{11}u_3(x_1)c(x_1) - (\partial_1\theta)^2| = Q_s(\partial_{11}u_3(x_1), \partial_1\theta(x_1)).$$

It is immediate to see that

$$c(x_1) = \begin{cases} (\partial_1\theta(x_1))^2/\partial_{11}u_3(x_1) & \text{if } |\partial_{11}u_3(x_1)| > |\partial_1\theta(x_1)|, \\ \partial_{11}u_3(x_1) & \text{if } |\partial_{11}u_3(x_1)| \le |\partial_1\theta(x_1)|. \end{cases}$$

Thus, $c \in L^2(0,L)$. Define

$$M := \begin{pmatrix} \partial_{11}u_3 & \partial_1\theta \\ \partial_1\theta & c \end{pmatrix},$$

and let $(\lambda^n) \subset C^\infty([0,L])$, $(p^n) \subset C^\infty([0,L];\mathbb{R}^2)$ be the sequences provided by Lemma 5.2.6. Precisely, we have $|p^n| \equiv 1$, $p^n \neq e_2$ in $[0,L]$ for every $n \in \mathbb{N}$, and defining $M^n := \lambda^n p^n \otimes p^n$, we have $M^n \rightharpoonup M$ in L^2 and

$$\int_0^L |M^n|^2\,dx \to \int_0^L |M|^2\,dx + 2\int_0^L |\det(M)|\,dx = I^{\mathrm{VKS}}(u_3,\theta).$$

Let R_h^n be the unique global solution of the Cauchy's problem

$$\begin{cases} \partial_1 X = \delta_h^{\gamma-1} X \partial_1 \tilde{A}^n & \text{in } [0,L], \\ X(0) = \exp(\delta_h^{\gamma-1}\tilde{A}^n(0)), \end{cases}$$

where

$$\tilde{A}^n := \begin{pmatrix} 0 & 0 & -\bar{M}_{11}^n \\ 0 & 0 & -\bar{M}_{12}^n \\ \bar{M}_{11}^n & \bar{M}_{12}^n & 0 \end{pmatrix},$$

and

$$\bar{M}^n(x_1) := \begin{pmatrix} \partial_1 u_3(0) & \theta(0) \\ \theta(0) & 0 \end{pmatrix} + \int_0^{x_1} M^n(s)\,ds.$$

We set $d_1^{h,n} := R_h^n e_1$ and $d_2^{h,n} := R_h^n e_2$, and we define

$$\tilde{y}_h^n := \begin{pmatrix} \delta_h^{2\gamma-2}u_1(0) \\ 0 \\ \delta_h^{\gamma-1}u_3(0) \end{pmatrix} + \int_0^{x_1} d_1^{h,n}(s)\,ds.$$

Let $M^n_h := \delta_h^{\gamma-1} M^n$ and $\lambda^n_h := \delta_h^{\gamma-1}\lambda^n$. We are in a position to apply Theorem 2.2.15, with the choices $y := \tilde{y}^n_h$, $d_2 := d_2^{h,n}$, $\lambda := \lambda^n_h$, and $p := p^n$, up to a continuous extension in a common neighbourhood of $[0, L]$. For some $\varepsilon_n > 0$, depending solely on n, there are isometric immersions

$$w^n_h \in W^{1,\infty}(\Omega_{\varepsilon_n}, \mathbb{R}^3) \cap W^{2,2}_{\text{iso}}(\Omega_{\varepsilon_n}; \mathbb{R}^3),$$

defined by the relation

$$w^n_h(\Phi^n(x_1, x_2)) = \tilde{y}^n_h(x_1) + x_2 \begin{pmatrix} d_1^{h,n}(x_1) & d_2^{h,n}(x_1) \end{pmatrix} (p^n)^\perp(x_1)$$

for $(x_1, x_2) \in (\Phi^n)^{-1}(\Omega_{\varepsilon_n}) \subset \Omega_{\eta_n}$, where $\eta_n > 0$. Moreover, w^n_h satisfies properties (a)–(c) of Theorem 2.2.15. Here

$$\Phi^n(x_1, x_2) := x_1 e_1 + x_2 (p^n)^\perp(x_1),$$

is an invertible bi-Lipschitz homeomorphism onto Ω_{ε_n}. For $h \le \varepsilon_n$, we define $T_h(x_1, x_2) := (x_1, hx_2)$ and

$$y^n_h := w^n_h \circ T_h \in W^{2,2}(\Omega; \mathbb{R}^3).$$

We start by showing that

$$E^\alpha_h(y^n_h) \xrightarrow{h\to 0} \int_0^L |M^n|^2 \, dx. \tag{5.34}$$

Since w^n_h is an isometric immersion, the elastic energy term vanishes. Indeed,

$$(\nabla_h y^n_h)^T \nabla_h y^n_h - \mathrm{Id} = (\nabla w^n_h \circ T_h)^T (\nabla w^n_h \circ T_h) - \mathrm{Id} = 0. \tag{5.35}$$

By Theorem 2.2.15–(b), we have

$$\nabla w^n_h(\Phi^n(x_1, x_2)) = \begin{pmatrix} d_1^{h,n}(x_1) & d_2^{h,n}(x_1) \end{pmatrix}. \tag{5.36}$$

In particular, defining $\nu^n_h := \partial_1 w^n_h \wedge \partial_2 w^n_h$, we get

$$\nu^n_h(\Phi^n(x_1, x_2)) = d_3^{h,n}(x_1).$$

Recall that Φ^n is invertible. To simplify the notation, let us write q^n in place of $(\Phi^n)^{-1}$. We get

$$\begin{aligned} \partial_1 w^n_h &= d_1^{h,n}(q^n_1), \\ \partial_2 w^n_h &= d_2^{h,n}(q^n_1), \\ \nu^n_h &= d_3^{h,n}(q^n_1). \end{aligned}$$

Differentiating, we obtain

$$\begin{aligned} \partial_{11} w^n_h &= \partial_1 d_1^{h,n}(q^n_1)\partial_1 q^n_1, \\ \partial_{12} w^n_h &= \partial_1 d_2^{h,n}(q^n_1)\partial_1 q^n_1, \\ \partial_{22} w^n_h &= \partial_1 d_2^{h,n}(q^n_1)\partial_2 q^n_1. \end{aligned}$$

Thus, by Theorem 2.2.15–(c),

$$\partial_{ij} w_h^n \cdot \nu_h^n = \delta_h^{\gamma-1} M_{1j}^n(q_1^n)\partial_i q_1^n, \qquad i,j = 1,2.$$

Note that

$$\nabla\Phi^n(x_1,x_2) = \begin{pmatrix} e_1 + x_2\partial_1(p^n)^\perp(x_1) & (p^n)^\perp(x_1)\end{pmatrix},$$

and so

$$\nabla\Phi^n(T_h) \to \begin{pmatrix} e_1 & (p^n)^\perp(x_1)\end{pmatrix}$$

pointwise as $h \to 0$. Note also that by continuity of q^n, we have

$$q^n(T_h) \to q^n(x_1,0) = x_1 e_1 \tag{5.37}$$

pointwise. Since

$$\nabla q^n = (\nabla\Phi^n)^{-1}(q^n)$$

and

$$(\nabla\Phi^n)^{-1} = \frac{1}{(p^n)_2^\perp}\begin{pmatrix} (p^n)_2^\perp & -(p^n)_1^\perp \\ 0 & 1 \end{pmatrix},$$

by (5.37) we get that

$$\begin{aligned} \nabla q_1^n(T_h) \to e_1^T(\nabla\Phi^n)^{-1}(x_1,0) &= \frac{1}{(p^n(x_1))_2^\perp}\begin{pmatrix} (p^n(x_1))_2^\perp & -(p^n(x_1))_1^\perp \end{pmatrix} \\ &= \frac{1}{p_1^n(x_1)}(p^n(x_1))^T \end{aligned}$$

pointwise as $h \to 0$, where we have used the fact that

$$(p^n)^\perp = \begin{pmatrix} p_2^n \\ -p_1^n \end{pmatrix}.$$

In particular, for $i,j = 1,2$,

$$\begin{aligned} \delta_h^{1-\gamma}\partial_{ij} w_h^n(T_h) \cdot \nu_h^n(T_h) &= M_{1j}^n(q_1^n(T_h))(\partial_i q_1^n)(T_h) \\ &\to \frac{1}{p_1^n(x_1)} M_{1j}^n(x_1) p_i^n(x_1) = M_{ij}^n(x_1) \end{aligned} \tag{5.38}$$

pointwise as $h \to 0$, where in the last equality we have used the fact that $M^n = \lambda^n p^n \otimes p^n$. Recall that, by Proposition 5.2.4, we have

$$\sum_{i,j=1}^{2} |\partial_{ij} w_h^n|^2 = \sum_{i,j=1}^{2} |\partial_{ij} w_h^n \cdot \nu_h^n|^2.$$

Thus, since $\nabla_h^2 y_h = \nabla^2 w_h^n \circ T_h$, by (5.38) and Dominated Convergence Theorem, we deduce

$$\delta_h^{2-\alpha}\int_\Omega |\nabla_h^2 y_h^n|^2\, dx = \delta_h^{2-\alpha}\sum_{i,j=1}^{2}\int_\Omega |\partial_{ij} w_h^n(T_h)\cdot \nu_h^n(T_h)|^2\, dx \to \int_0^L |M^n|^2\, dx.$$

Recalling (5.35), we conclude the proof of (5.34).

We move now to the proof of the convergence of the rescaled displacements. Define

$$\begin{aligned} u^n_{h,1} &:= \delta_h^{2-2\gamma}(y^n_{h,1} - x_1), \\ u^n_{h,2} &:= h\delta_h^{2-2\gamma}(y^n_{h,2} - hx_2), \\ u^n_{h,3} &:= \delta_h^{1-\gamma} y^n_{h,3}, \\ A^n_h &:= \delta_h^{1-\gamma}(\nabla_h y_h - \mathrm{Id}_{3\times 2}). \end{aligned}$$

Recall that, by the boundedness of the energy (see Proposition 5.3.1), $u^n_{h,1}$, $u^n_{h,2}$, $u^n_{h,3}$, and A^n_h converge, up to subsequences, in their respective spaces, that is

$$\begin{aligned} u^n_{h,1} &\rightharpoonup u^n_1 && \text{in } W^{1,2}(\Omega), \\ u^n_{h,2} &\rightharpoonup 0 && \text{in } W^{2,2}(\Omega), \\ u^n_{h,3} &\rightharpoonup u^n_3 && \text{in } W^{2,2}(\Omega), \\ A^n_h &\rightharpoonup A^n && \text{in } W^{1,2}(\Omega;\mathbb{R}^{3\times 2}). \end{aligned}$$

Thus, it is sufficient to identify u^n_1, u^n_3, and A^n. Arguing as in proof of Theorem 5.3.7, we have $R^n_h \in SO(3)$ everywhere in $[0, L]$, and

$$\begin{aligned} R^n_h = \mathrm{Id} + \delta_h^{\gamma-1}\tilde{A}^n + \delta_h^{2\gamma-2}\int_0^{x_1} \tilde{A}^n(s)\partial_1\tilde{A}^n(s)\,ds \\ + \frac{1}{2}\delta_h^{2\gamma-2}(\tilde{A}^n)^2(0) + O(\delta^{3\gamma-3}), \end{aligned} \tag{5.39}$$

where the big-O notation is used with respect to the convergence $h \to 0$. Thus

$$\begin{aligned} (R^n_h)_{11} &= 1 - \delta_h^{2\gamma-2}\int_0^{x_1} \bar{M}^n_{11} M^n_{11}\,ds - \frac{1}{2}\delta_h^{2\gamma-2}(\partial_1 u_3)^2(0) + O(\delta_h^{3\gamma-3}) \\ &= 1 - \frac{1}{2}\delta_h^{2\gamma-2}(\bar{M}^n_{11})^2 + O(\delta_h^{3\gamma-3}), \end{aligned} \tag{5.40}$$

$$(R^n_h)_{12} = O(\delta_h^{2\gamma-2}), \tag{5.41}$$

$$(R^n_h)_{21} = O(\delta_h^{2\gamma-2}), \tag{5.42}$$

$$(R^n_h)_{22} = 1 + O(\delta_h^{2\gamma-2}). \tag{5.43}$$

By (5.36) and (5.40)–(5.41), we have

$$\begin{aligned} \partial_1(w^n_h)_1 &= (R^n_h(q^n_1))_{11} = 1 - \frac{1}{2}\delta_h^{2\gamma-2}(\bar{M}^n_{11}(q^n_1))^2 + O(\delta_h^{3\gamma-3}), \\ \partial_2(w^n_h)_1 &= (R^n_h(q^n_1))_{12} = O(\delta_h^{2\gamma-2}). \end{aligned}$$

Thus,

$$\begin{aligned} \partial_1 u^n_{h,1} &= \delta_h^{2-2\gamma}((\partial_1 w^n_h(T_h))_1 - 1) = -\frac{1}{2}(M^n_{11}(q^n_1(T_h)))^2 + O(\delta_h^{\gamma-1}), \\ \partial_2 u^n_{h,1} &= \delta_h^{2-2\gamma}(h(\partial_2 w^n_h(T_h))_1) = O(h). \end{aligned}$$

Since $T_h \to x_1 e_1$ pointwise as $h \to 0$, by the continuity of both q^n and $\bar{M}^n$, it follows that

$$\begin{aligned}
\partial_1 u^n_{h,1} &\to -\frac{1}{2}(\bar{M}^n_{11}(x_1))^2, \\
\partial_2 u^n_{h,1} &\to 0.
\end{aligned}$$

In particular,

$$u^n_{h,1} \rightharpoonup c_1 - \frac{1}{2}\int_0^{x_1} (\bar{M}^n_{11}(s))^2\, ds = u^n_1 \qquad \text{in } W^{2,2}(\Omega).$$

Since by Theorem 2.2.15–(a) we have

$$u^n_{h,1}(0,0) = \delta_h^{2-2\gamma}(w^n_h)_1(0,0) = \delta_h^{2-2\gamma}\tilde{y}^n_h(0) = u_1(0)$$

we have $c_1 = u_1(0)$. By (5.42)–(5.43) and (5.36), we get

$$\begin{aligned}
(\partial_1 w^n_h)_3 &= (R^n_h(q^n_1))_{31} = \delta_h^{\gamma-1}\bar{M}^n_{11}(q^n_1) + O(\delta_h^{2\gamma-2}), \\
(\partial_2 w^n_h)_2 &= (R^n_h(q^n_1))_{32} = O(\delta_h^{\gamma-1}).
\end{aligned}$$

Thus, arguing as before

$$\begin{aligned}
\partial_1 u^n_{h,2} &= \delta_h^{1-\gamma}((\partial_1 w^n_h(T_h))_1) = \bar{M}^n_{11}(q^n_1(T_h)) + O(\delta_h^{\gamma-1}) \to \bar{M}^n_{11}(x_1), \\
\partial_2 u^n_{h,2} &= \delta_h^{1-\gamma}(h(\partial_2 w^n_h(T_h))_1) = O(h) \to 0.
\end{aligned}$$

It follows that

$$u^n_3(x_1) = c_3 + \int_0^{x_1} \bar{M}^n_{11}(s)\, ds,$$

where, arguing as before, we get $c_3 = u_3(0)$. Lastly, by (5.39), we get

$$\nabla w^n_h = \mathrm{Id} + \delta_h^{\gamma-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \bar{M}^n_{11}(q^n_1) & \bar{M}^n_{12}(q^n_1) \end{pmatrix} + O(\delta_h^{2\gamma-2}).$$

Thus

$$\begin{aligned}
A^n_h = \delta_h^{1-\gamma}(\nabla w_h(T_h) - \mathrm{Id}) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \bar{M}^n_{11}(q^n_1(T_h)) & \bar{M}^n_{12}(q^n_1(T_h)) \end{pmatrix} + O(\delta_h^{\gamma-1}) \\
&\to \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \bar{M}^n_{11}(x_1) & \bar{M}^n_{12}(x_1) \end{pmatrix} = A^n.
\end{aligned}$$

Since $M^n \rightharpoonup M^n$ in L^2, by definition of $\bar{M}^n$ we immediately deduce that

$$\begin{aligned}
u^n_1 &\to u_1(0) - \frac{1}{2}\int_0^{x_1} \left(\partial_1 u_3(0) + \int_0^s \partial_{11} u_3(z)\, dz\right)^2 ds \\
&= u_1(0) - \frac{1}{2}\int_0^{x_1} (\partial_1 u_3(s))^2\, ds = u_1(0) + \int_0^{x_1} \partial_1 u_1(s)\, ds = u_1.
\end{aligned}$$

where the convergence is in $W^{1,2}(\Omega)$. Similarly,

$$u_3^n \to u_3(0) + \int_0^{x_1} \Big(\partial_1 u_3(0) + \int_0^s \partial_{11} u_3(z)\, dz\Big)\, ds = u_3(0) + \int_0^{x_1} \partial_1 u_3(s)\, ds = u_3$$

in $W^{2,2}(\Omega)$. Arguing similarly, we deduce that $A^n \to A$ in $W^{1,2}$. We are finally in a position to apply a diagonal argument and construct a diagonal sequence y_h such that convergences (i)–(v) hold, concluding the proof. □

II

Problems motivated by dislocation theory

6 Notation and mathematical preliminaries

6.1 Special functions

We recall here the definition and the main properties of some special functions that we use in Chapter 7. For further details we refer to the monographs [Erd53; Leb65].

The Gamma and the Beta functions

The Gamma function is defined as

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt$$

for $z \in \mathbb{C}$ with $\Re(z) > 0$. It can be extended by analytic continuation to the whole complex plane, except at non-positive integers. We use the following notable properties of Γ:

(i) $\Gamma(z+1) = z\Gamma(z)$ for every $z \in \mathbb{C}$, $z \neq 0, -1, -2, \dots$, and in particular $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$,

(ii) $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ for every $z \in \mathbb{C}$, $z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots$,

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,

(iv) $\Gamma(x+\alpha) \sim \Gamma(x)x^\alpha$ as $x \to \infty$ for $\alpha \in \mathbb{C}$.

The Beta function can be defined in terms of the Gamma function as

$$B(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y \in \mathbb{C}$ with $\Re(x) > 0$ and $\Re(y) > 0$. It easily follows that B is symmetric.

The hypergeometric function

Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \neq 0, -1, -2, \dots$. The hypergeometric function ${}_2F_1$ is defined as the power series

$$ {}_2F_1(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n $$

for $z \in \mathbb{C}$ with $|z| < 1$. Here $(\lambda)_n$ denotes the Pochhammer's symbol, namely

$$ (\lambda)_0 := 1, \qquad (\lambda)_n := \lambda(\lambda+1) \cdot \dots \cdot (\lambda+n-1) \quad \text{for } n \in \mathbb{N}. $$

If $-1 < \Re(\gamma - \alpha - \beta)$, then the series converges for $|z| \leq 1$, except at the point $z = 1$. The behaviour near the point $z = 1$ depends on the parameters α, β, and γ. More precisely, if $\Re(\gamma - \alpha - \beta) > 0$, the series extends continuously also at $z = 1$ (see [Erd53, Section 2.1.3–(14)] or [Leb65, Section 9.3]), and we have

$$ \lim_{z \to 1^-} {}_2F_1(\alpha, \beta; \gamma; z) = {}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}. \tag{6.1} $$

If $\gamma = \alpha + \beta$, from [Erd53, Section 2.3.1–(2)] we can deduce that

$$ \lim_{z \to 1^-} \frac{{}_2F_1(\alpha, \beta; \alpha + \beta; z)}{-\log(1-z)} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}, \tag{6.2} $$

while if $\Re(\gamma - \alpha - \beta) < 0$, from (6.1) and [Erd53, Section 2.1.4–(23)] we have

$$ \lim_{z \to 1^-} \frac{{}_2F_1(\alpha, \beta; \gamma; z)}{(1-z)^{\gamma - \alpha - \beta}} = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}. \tag{6.3} $$

Note that if α or β is a non-positive integer, then the hypergeometric function reduces to a polynomial in z. In particular, it holds that

$$ {}_2F_1(0, \beta; \gamma; z) = 1 \tag{6.4} $$

and

$$ {}_2F_1(-1, \beta; \gamma; z) = 1 - \frac{\beta}{\gamma} z. \tag{6.5} $$

The Appell function of the fourth kind

Let $\alpha, \beta, \gamma, \gamma' \in \mathbb{C}$ with $\gamma, \gamma' \neq 0, -1, -2, \dots$. The Appell function of the fourth kind F_4 is defined as the double power series

$$ F_4(\alpha, \beta; \gamma, \gamma'; x, y) := \sum_{n,m=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n n! m!} x^m y^n $$

for $x, y \in \mathbb{C}$ with $\sqrt{|x|} + \sqrt{|y|} < 1$. Since F_4 is analytic in its domain, it follows that

$$ \lim_{x \to 0} F_4(\alpha, \beta; \gamma, \gamma'; x, y) = {}_2F_1(\alpha, \beta; \gamma'; y) \tag{6.6} $$

for every $y \in \mathbb{C}$ with $|y| < 1$.

The Bessel function of the first kind

Let $\nu \in \mathbb{C}$. The Bessel function of the first kind of order ν is defined as

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{\Gamma(n+1)\Gamma(n+\nu+1)}$$

for $x \in \mathbb{C} \setminus (-\infty, 0)$. We recall the asymptotic behavior of J_ν at 0 and $+\infty$:

$$J_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(1+\nu)}, \qquad \text{as } x \to 0^+, \tag{6.7}$$

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right), \qquad \text{as } x \to +\infty, \tag{6.8}$$

see [Leb65, Section 5.16]. Moreover, we use the following identity

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos(z) \qquad \text{for } z \neq 0, \tag{6.9}$$

that can be found in [Leb65, Equation (5.8.4)].

6.1.1 Integral formulas

We recall here some integral formulas involving the special functions we introduced.

Formula 6.1.1 [GR07, Formula 3.621–5]. *We have*

$$\int_0^{\frac{\pi}{2}} \sin^{\mu-1}(x) \cos^{\mu-1}(x)\, dx = \frac{1}{2} B\Big(\frac{\mu}{2}, \frac{\nu}{2}\Big), \qquad \Re(\mu), \Re(\nu) > 0.$$

Formula 6.1.2 [GR07, Formula 3.665–2]. *For $\Re(\mu) > 0$ and $|a| < 1$ we have*

$$\int_0^{\pi} \frac{\sin^{2\mu-1}(x)}{(1 + 2a\cos(x) + a^2)^\nu}\, dx = B\Big(\mu, \frac{1}{2}\Big) {}_2F_1\Big(\nu, \nu - \mu + \frac{1}{2}; \mu + \frac{1}{2}; a^2\Big).$$

Formula 6.1.3 [GR07, Formula 6.567–1]. *For $b > 0$, $\Re(\nu) > -1$, and $\Re(\rho) > -1$ we have*

$$\int_0^1 x^{\nu+1}(1-x^2)^\rho J_\nu(bx)\, dx = \frac{2^\rho \Gamma(\rho+1)}{b^{\rho+1}} J_{\nu+\rho+1}(b).$$

Formula 6.1.4 [Bai36, Formula 7.1]. *Provided that*

$$\Re(\lambda + \mu + \nu + \rho) > 0, \quad \Re(\lambda) < \frac{5}{2}, \quad c > |a| + |b|, \tag{6.10}$$

we have

$$\begin{aligned}
&\int_0^\infty x^{\lambda-1} J_\nu(ax) J_\mu(bx) J_\rho(cx)\, dx \\
&= \frac{2^{\lambda-1} a^\nu b^\mu \Gamma(\frac{1}{2}(\lambda+\mu+\nu+\rho))}{c^{\lambda+\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(1-\frac{1}{2}(\lambda+\mu+\nu-\rho))} \\
&\quad \times F_4\Big(\frac{1}{2}(\lambda+\mu+\nu-\rho), \frac{1}{2}(\lambda+\mu+\nu+\rho); \mu+1, \nu+1; \frac{b^2}{c^2}, \frac{a^2}{c^2}\Big).
\end{aligned}$$

Formula 6.1.5 [GR07, Formula 3.251–1]. *For $\Re(\mu) > 0$, $\Re(\nu) > 0$, and $\lambda > 0$ we have*

$$\int_0^1 x^{\mu-1}(1-x^\lambda)^{\nu-1}\, dx = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \nu\right).$$

6.2 Fourier's transform

We recall here the definition of the Fourier's transform of a tempered distribution, and a few properties that are useful for our purposes.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz class of rapidly decreasing functions, namely

$$\mathcal{S}(\mathbb{R}^d) := \{\phi \in C^\infty(\mathbb{R}^d) : \sup |x^\beta \partial^\alpha \phi(x)| < \infty, \forall\, \alpha, \beta \text{ multi-index}\},$$

where for a multi-index $\beta := (\beta_1, \dots, \beta_d)$ we write $x^\beta := \prod_{i=1}^d x^{\beta_i}$. We endow $\mathcal{S}(\mathbb{R}^d)$ with the Frèchet's topology induced by the seminorms

$$[\phi]_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\beta \partial^\alpha \phi|, \qquad \alpha, \beta \text{ multi-indices.}$$

Let $\mathcal{S}'(\mathbb{R}^d)$ be the dual space of $\mathcal{S}(\mathbb{R}^d)$, that is, the space of tempered distributions. Following [Fol92], we define the Fourier's transform $\widehat{f}$ of a function $f \in \mathcal{S}(\mathbb{R}^d)$ as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx \tag{6.11}$$

for $\xi \in \mathbb{R}^d$. Note that if $f \in \mathcal{S}(\mathbb{R}^d)$ then $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$. If T is a tempered distribution, its Fourier's transform can be defined by duality, namely

$$\langle \widehat{T}, \varphi \rangle := \langle T, \widehat{\varphi} \rangle \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^d).$$

If $f \in L^1(\mathbb{R}^d)$, then clearly $f \in \mathcal{S}'(\mathbb{R}^d)$, its distributional Fourier's transform $\widehat{f}$ can be identified with a $C^0(\mathbb{R}^d)$ function, and it coincides with (6.11). A result that is crucial for our arguments in Chapter 7 is the Fourier Inversion Theorem, that we recall here.

Theorem 6.2.1 (Fourier Inversion Theorem). *Let $f \in L^1(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$. Suppose that $\widehat{f} \in L^1(\mathbb{R}^d)$. Then*

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi \qquad \forall\, x \in \mathbb{R}^d. \tag{6.12}$$

For a proof, see for example [Fol92, p. 244]. As a remark, observe that the continuity of f is a necessary condition that follows directly from (6.12).

The Fourier Inversion Theorem also holds for tempered distributions. Precisely, if $T \in \mathcal{S}'(\mathbb{R}^d)$, then

$$\langle T, \phi \rangle = \frac{1}{(2\pi)^d} \langle \widehat{T}, \widehat{\phi}_- \rangle \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^d), \tag{6.13}$$

where $\phi_-(x) := \phi(-x)$. However, equality (6.13) holds only in the sense of distributions, while for our purposes we need a pointwise equality as in (6.12). In the next remark, we point out that Theorem 6.2.1 still holds under slightly milder integrability conditions for f.

Remark 6.2.2. Theorem 6.2.1 continues to hold (with equality a.e. in Equation (6.12)) if $f \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{\text{loc}}(\mathbb{R}^d)$ and $\widehat{f} \in L^1(\mathbb{R}^d)$. Firstly, note that the Fourier's transform of f has to be computed in the distributional sense. However, the Fourier's transform of $\widehat{f}$ coincides with the classical one. Moreover, by the Fourier Inversion Theorem for tempered distributions and Fubini's Theorem, we have that for any test function $\phi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\cdot\phi(x)\,dx = \langle f,\phi\rangle &= \frac{1}{(2\pi)^d}\langle \widehat{f},\widehat{\phi}_-\rangle \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\xi)\cdot\int_{\mathbb{R}^d}\phi(-x)e^{-ix\cdot\xi}\,dx\,d\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\phi(x)\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi\,dx.\end{aligned}$$

Then, we can conclude by the Fundamental Lemma of the Calculus of Variations. Note that the hypothesis $\widehat{f} \in L^1(\mathbb{R}^d)$ is crucial to apply Fubini's Theorem. Moreover, as a by-product, f must have a continuous representative.

If f is a tempered distribution that can be identified with an L^1 function whose Fourier's transform is L^1, the Parseval formula holds.

Lemma 6.2.3 (Parseval's Formula)**.** *Let $f, g \in L^1(\mathbb{R}^d)$ be such that $\widehat{f}, \widehat{g} \in L^1(\mathbb{R}^d)$. Then $f, g, \widehat{f}, \widehat{g} \in L^2(\mathbb{R}^d)$ and we have*

$$\int_{\mathbb{R}^d} f(x)g(x)\,dx = \int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}\widehat{g}(\xi)\,d\xi,$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Proof. Note that, by the Fourier Inversion Theorem 6.2.1, both f and g are continuous, whereas $\widehat{f}$ and $\widehat{g}$ are continuous by definition. Thus, to show that $f, g, \widehat{f}$, and $\widehat{g}$ belong to $L^2(\mathbb{R}^d)$, it is sufficient to prove that they tend to zero at infinity. Without loss of generality, we show the argument for f. Since $\widehat{f} \in L^1(\mathbb{R}^d)$, by density of $\mathcal{S}(\mathbb{R}^d)$ there is a sequence $(\varphi^n) \subset \mathcal{S}(\mathbb{R}^d)$ such that $\varphi^n \to \widehat{f}$ in L^1. Defining $\varphi^n_-(x) := \varphi^n(-x)$, by the Fourier Inversion Theorem 6.2.1 we have

$$\sup_{x\in\mathbb{R}^d} |f(x) - \widehat{\varphi}^n_-(x)| \le \|\widehat{f} - \varphi^n\|_{L^1} \to 0,$$

that gives the uniform convergence of $\widehat{\varphi}^n_-$ to f. Since $\widehat{\varphi}^n_- \in \mathcal{S}(\mathbb{R}^d)$, we get $f \to 0$ at infinity.

It follows that $f, g, \widehat{f}, \widehat{g} \in L^2(\mathbb{R}^d)$. Then, we can conclude by the following chain of equalities:

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)g(x)\,dx = \int_{\mathbb{R}^d}\overline{f(x)}g(x)\,dx &= \int_{\mathbb{R}^d}\left[\overline{\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi}\right]g(x)\,dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}e^{-ix\cdot\xi}g(x)\,d\xi\,dx = \int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}\widehat{g}(\xi)\,d\xi,\end{aligned}$$

where we have applied the Fourier Inversion Theorem 6.2.1 and Fubini's Theorem.

□

Remark 6.2.4. Lemma 6.2.3 also holds requiring different regularities for f and g. This is clear observing the proof. Indeed, we just need to grant that the L^2 inner product is well-defined, that the hypothesis of the Fourier Inversion Theorem are satisfied by one of the functions, and that we can apply Fubini's Theorem. For example, suppose that $f \in L^1_{\text{loc}}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d)$ with compact support and $\widehat{f}, \widehat{g} \in L^1(\mathbb{R}^d)$. Then, the integral

$$\int_{\mathbb{R}^d} f(x)g(x)\,dx$$

is well-defined, by Remark 6.2.2 the Fourier Inversion Theorem holds for f, and since both $\widehat{f}, g \in L^1(\mathbb{R}^d)$ and $e^{-ix\cdot\xi} \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ we can apply Fubini's Theorem to change the order of integration.

In Chapter 7, we compute various Fourier's transforms. If f is a radial function, we resort to the following formula.

Formula 6.2.5. *If f is a radial function, that is, $f(x) = f_0(|x|)$ for some f_0, then so is $\widehat{f}$ and we have*

$$\widehat{f}(\xi) = \frac{(2\pi)^{\frac{d}{2}}}{|\xi|^{\frac{d}{2}-1}} \int_0^\infty f_0(r) J_{\frac{d}{2}-1}(r|\xi|) r^{\frac{d}{2}}\,dr. \tag{6.14}$$

We refer to [Fol92, eq. (7.38)] for a proof. Formula 6.2.5 can be applied also to more general objects with a radial symmetry. As an example, consider $\mu := \mathcal{H}^{d-1} \llcorner \partial B_1$. One may be tempted to apply (6.14) with $f_0(r) := \delta_1(r)$, interpreting the integral in a distributional sense. Indeed, we have that

$$\widehat{\mu}(\xi) = \frac{(2\pi)^{\frac{d}{2}}}{|\xi|^{\frac{d}{2}-1}} J_{\frac{d}{2}-1}(|\xi|). \tag{6.15}$$

This can be obtained, for example, following the computation in [Gra08, Appendix B.4].

Another class of tempered distributions whose properties are preserved by the Fourier's transform are the homogeneous ones. We say that a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ is α-homogeneous if, for every test function $\phi \in C_c^\infty$ and for every $\lambda \in \mathbb{R}$ we have

$$\langle T, \phi_\lambda \rangle = \lambda^{-\alpha} \langle T, \phi \rangle,$$

where $\phi_\lambda(x) := \lambda^d \phi(\lambda x)$. The Fourier's transform maps α-homogeneous tempered distributions to $(-d-\alpha)$-homogeneous tempered distributions. Indeed, we have the following Lemma.

Lemma 6.2.6. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$ be an α-homogeneous tempered distribution. Then $\widehat{T}$ is a $(-d-\alpha)$-homogeneous tempered distribution.*

Proof. Set $\phi_\lambda(x) := \lambda^d \phi(\lambda x)$ for some $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$. We have

$$\begin{aligned}
\widehat{\phi}_\lambda(\xi) &= \int_{\mathbb{R}^d} \phi_\lambda(x) e^{-ix\cdot\xi}\,dx = \lambda^d \int_{\mathbb{R}^d} \phi(\lambda x) e^{-ix\cdot\xi}\,dx = \int_{\mathbb{R}^d} \phi(y) e^{-iy\cdot\frac{\xi}{\lambda}}\,dy \\
&= \widehat{\phi}\left(\frac{\xi}{\lambda}\right) = \lambda^d (\widehat{\phi})_{\lambda^{-1}}(\xi).
\end{aligned}$$

Then,

$$\langle \widehat{T}, \phi_\lambda \rangle = \langle T, \widehat{\phi_\lambda} \rangle = \lambda^d \langle T, (\widehat{\phi})_{\lambda^{-1}} \rangle = \lambda^{d+\alpha} \langle T, \widehat{\phi} \rangle = \lambda^{d+\alpha} \langle \widehat{T}, \phi \rangle.$$

□

In Chapter 7, we consider an interaction kernel defining a $(-\alpha)$-homogeneous tempered distribution, with $\alpha \in (0, d)$. It follows that its Fourier's transform is $(\alpha - d)$-homogeneous. It is useful to note that in that case, it is enough to consider the restriction of the Fourier's transform to $\mathbb{R}^d \setminus \{0\}$. Recall that given a distribution $T \in \mathcal{D}'(\mathbb{R}^d)$, its restriction to an open set $A \subset \mathbb{R}^d$ is a distribution in $\mathcal{D}'(A)$.

Lemma 6.2.7. *Let T be a $(-\alpha)$-homogeneous tempered distribution with $\alpha \in (0, d)$. Suppose that $T|_{\mathbb{R}^d \setminus \{0\}}$ can be identified with a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$. Then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $T = f$, in the sense that*

$$\langle T, \phi \rangle = \int_{\mathbb{R}^d} f(x)\phi(x)\, dx \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d).$$

Proof. Note that f is $(-\alpha)$-homogeneous in the classical sense. In particular, since $\alpha \in (0, d)$, it is integrable near the origin, thus $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Define the distribution $G := T - f$, that is

$$\langle G, \phi \rangle := \langle T, \phi \rangle - \int_{\mathbb{R}^d} f(x)\phi(x)\, dx \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d).$$

Note that $G = 0$ on $\mathbb{R}^d \setminus \{0\}$. In particular, the support of G is contained in the singleton $\{0\}$. By a classical result on distributions (see for example [GS16b, Chapter II Section 4.5]), G has the following form:

$$G = \sum_{|\beta| \le m} \lambda_\beta D^\beta \delta_0,$$

for some $m \in \mathbb{N}$ and $\lambda_\beta \in \mathbb{R}$, where β is a multi-index. Note that G is $(-\alpha)$-homogeneous. Since $D^\beta \delta_0$ is $(-d - |\beta|)$-homogeneous and homogeneous distributions with different homogeneity are linearly independent (see, for example, [GS16a, Chapter I Section 3.11]) it follows that $\lambda_\beta = 0$ for all $|\beta| \le m$, concluding the proof. □

An example of homogeneous tempered distribution that we use in Chapter 7, is $p_k(x)|x|^{-s-k}$, where p_k is a harmonic homogeneous polynomial of degree k. In the next Lemma, we compute its Fourier's transform following [Ste71].

Lemma 6.2.8. *Let p_k be a homogeneous harmonic polynomial of degree $k \ge 0$. Let $s \in (0, d)$. Then, the Fourier's transform of $p_k(x)|x|^{-s-k}$ is the map*

$$\xi \mapsto (-i)^k 2^{d-s} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{k+d-s}{2})}{\Gamma(\frac{k+s}{2})} \frac{p_k(\xi)}{|\xi|^{k+d-s}}. \tag{6.16}$$

For a proof of Lemma 6.2.8, we refer to [Ste71, Chapter II Equation (33)]. Note that Equation (33) holds for $k \geq 0$, even if it is stated in [Ste71, Theorem 5], only for $k \geq 1$. We also point out that [Ste71] uses a slightly different definition of Fourier's transform, thus the coefficients in (6.16) are different from the ones in [Ste71, Equation (33)].

In the rest of the section we recall a few properties of the Fourier's transform of Radon measures. Let $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ be a finite Radon measure in $\mathbb{R}^d$ with compact support. It is easy to see that $\mu \in \mathcal{S}'(\mathbb{R}^d)$. By definition of Fourier's transform and an application of Fubini's Theorem

$$\langle \widehat{\mu}, \phi \rangle = \langle \mu, \widehat{\phi} \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) e^{-ix\cdot\xi}\, dx\, d\mu(\xi) = \int_{\mathbb{R}^d} \phi(x) \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\, d\mu(\xi)\, dx.$$

Since the map

$$x \mapsto \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\, d\mu(\xi)$$

is smooth by Dominated Convergence Theorem, we have $\widehat{\mu} \in C^\infty(\mathbb{R}^d)$. Note that a stronger result holds in a more general setting: the Fourier's transform of a compactly supported distribution is analytic (see [Hör76, Theorem 1.7.5]).

Recall that given two distributions $F, T \in \mathcal{D}'(\mathbb{R}^d)$, where T has compact support, the convolution $F * T$ is well-defined as an element of $\mathcal{D}'(\mathbb{R}^d)$ (see, for example, [Hör76, Definition 1.6.2]). Precisely, it is the unique distribution $G \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$F * (T * \phi) = G * \phi \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d).$$

If F happens to be also a tempered distribution, then $F * T \in \mathcal{S}'(\mathbb{R}^d)$. In the last case, it holds that $\widehat{F * T} = \widehat{F}\,\widehat{T}$, see [Hör76, Theorem 1.7.6]. Note that the product $\widehat{F}\widehat{T}$ is well defined as a distribution, since $\widehat{T} \in C^\infty(\mathbb{R}^d)$.

6.3 Spherical harmonics and Sobolev spaces on sphere

We recall here some results on spherical harmonics and on Sobolev spaces on the sphere. As it is customary, we write $\mathbb{S}^{d-1}$ for the $(d-1)$-dimensional sphere in $\mathbb{R}^d$. We use $\Delta_{\mathbb{S}^{d-1}}$ (or $\Delta_{\mathbb{S}}$ if there is no ambiguity on the dimension) to denote the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$. Recall that $\Delta_{\mathbb{S}^{d-1}}$ can be defined as

$$\Delta_{\mathbb{S}^{d-1}} f(x) := \Delta g(x), \tag{6.17}$$

where Δ is the standard Euclidean Laplacian and $g(x) := f(x/|x|)$.

A spherical harmonic of degree n, usually denoted as Y_n, is the restriction to the sphere of a harmonic homogeneous polynomial of degree n. Despite its name, Y_n is not harmonic on the sphere in the sense of the Laplace–Beltrami operator, meaning that $\Delta_{\mathbb{S}^{d-1}} Y_n \neq 0$. This is clear by identity (6.17). However, Y_n is an eigenfunction of $-\Delta_{\mathbb{S}^{d-1}}$ with eigenvalue $n(n+d-2)$, that is

$$-\Delta_{\mathbb{S}^{d-1}} Y_n = n(n+d-2) Y_n. \tag{6.18}$$

Clearly, spherical harmonics are continuous functions on $\mathbb{S}^{d-1}$. In particular, since $\mathbb{S}^{d-1}$ is compact, they are bounded. In Chapter 7, we use the following explicit bound, depending on n and d.

Lemma 6.3.1. *Any spherical harmonic Y_n of degree n satisfies*

$$\|Y_n\|_{L^\infty} \leq \sqrt{\Gamma\left(\frac{d}{2}\right)\frac{N(n,d)}{2\pi^{\frac{d}{2}}}}\|Y_n\|_{L^2},$$

where $N(n,d)$ is the number of linearly independent homogeneous harmonic polynomials of degree n in d variables.

A proof can be found in [EF14, Proposition 4.16]. Next, we recall the explicit expression of $N(n,d)$.

Lemma 6.3.2. *We have*

$$N(n,d) = \frac{2n+d-2}{n}\binom{n+d-3}{n-1} = \frac{2n+d-2}{n}\frac{\Gamma(n+d-2)}{\Gamma(n)\Gamma(d-1)}.$$

We refer to [EF14, Theorem 4.4] for a proof.

By an application of the Gram–Schmidt algorithm, we can construct a set $\{Y_n^i\}_{i=1,\dots,N(n,d)}$ of spherical harmonics of degree n that are orthonormal in the L^2 sense, i.e.,

$$\int_{\mathbb{S}^{d-1}} Y_n^i Y_n^j \, d\mathcal{H}^{d-1} = \delta_{ij},$$

where δ_{ij} is the Kronecker's delta and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff's measure. Moreover, the set

$$\{Y_n^i : n \in \mathbb{N},\, i = 1,\dots,N(n,d)\}$$

is an orthonormal basis of $L^2(\mathbb{S}^{d-1})$.

We move now to the definition of Sobolev's spaces on the sphere. Given a function $f \in L^2(\mathbb{S}^{d-1})$, where L^2 is to be intended with respect to the $(d-1)$-dimensional Hausdorff's measure, we can write f in spherical harmonics as

$$f = \sum_{n=0}^{\infty}\sum_{i=1}^{N(n,d)} \lambda_n^i Y_n^i. \tag{6.19}$$

By Parseval's identity,

$$\|f\|_{L^2}^2 = \sum_{n=0}^{\infty}\sum_{i=1}^{N(n,d)} (\lambda_n^i)^2.$$

Inspired by (6.18), we define the action of the fractional Laplace–Beltrami operator on f as

$$(-\Delta_{\mathbb{S}^{d-1}})^{\alpha} f := \sum_{n=0}^{\infty}\sum_{i=1}^{N(n,d)} [n(n+d-2)]^{\alpha}\lambda_n^i Y_n^i. \tag{6.20}$$

Clearly, the series on the right-hand side of (6.20) might not be convergent. We define the space $W^{\alpha,2}(\mathbb{S}^{d-1})$ as the set of functions $f \in L^2(\mathbb{S}^{d-1})$ such that

$$(-\Delta_{\mathbb{S}^{d-1}})^{\frac{\alpha}{2}} f \in L^2(\mathbb{S}^{d-1}).$$

Decomposing $f \in L^2(\mathbb{S}^{d-1})$ as in (6.19), by Parseval's identity we equivalently have that $f \in W^{\alpha,2}(\mathbb{S}^{d-1})$ if and only if

$$\sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} [n(n+d-2)]^{\alpha} (\lambda_n^i)^2 < \infty.$$

We endow $W^{\alpha,2}(\mathbb{S}^{d-1})$ with the norm

$$\| \cdot \|_{W^{\alpha,2}} := \| \cdot \|_{L^2} + \|(-\Delta_{\mathbb{S}^{d-1}})^{\frac{\alpha}{2}}(\cdot)\|_{L^2}.$$

Theorem 6.3.3. *The space $W^{\alpha,2}(\mathbb{S}^{d-1})$ is continuously embedded in $C^0(\mathbb{S}^{d-1})$ for $\alpha > (d-1)/2$.*

Proof. It is sufficient to show that

$$\|f\|_{L^\infty} \leq C\|f\|_{W^{\alpha,2}}. \tag{6.21}$$

Indeed, by definition, any function in $W^{\alpha,2}(\mathbb{S}^{d-1})$ can be written in spherical harmonics as

$$f = \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} \lambda_n^i Y_n^i,$$

with

$$\sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} (\lambda_n^i)^2 < \infty, \qquad \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} [n(n+d-2)]^{\alpha} (\lambda_n^i)^2 < \infty.$$

Since spherical harmonics are continuous, f is the $W^{\alpha,2}$-limit of continuous functions, and by (6.21) the limit is also uniform. Thus, f is continuous and by (6.21) the embedding is continuous. To prove (6.21), we proceed explicitely. By Lemma 6.3.1 and Hölder's inequality, we have

$$\begin{aligned}
\|f\|_{L^\infty} &\leq \sum_{n=0}^{\infty} \left\| \sum_{i=1}^{N(n,d)} \lambda_n^i Y_n^i \right\|_{L^\infty} \leq \sum_{n=0}^{\infty} \sqrt{\Gamma\left(\frac{d}{2}\right) \frac{N(n,d)}{2\pi^{\frac{d}{2}}}} \left\| \sum_{i=1}^{N(n,d)} \lambda_n^i Y_n^i \right\|_{L^2} \\
&= \sum_{n=0}^{\infty} \sqrt{\Gamma\left(\frac{d}{2}\right) \frac{N(n,d)}{2\pi^{\frac{d}{2}}}} \left(\sum_{i=1}^{N(n,d)} |\lambda_n^i|^2 \right)^{\frac{1}{2}} \\
&\leq C \sum_{n=0}^{\infty} \frac{\sqrt{N(n,d)}}{[n(n+d-2)]^{\frac{\alpha}{2}}} [n(n+d-2)]^{\frac{\alpha}{2}} \left(\sum_{i=1}^{N(n,d)} |\lambda_n^i|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{n=0}^{\infty} \frac{N(n,d)}{[n(n+d-2)]^{\alpha}} \right)^{\frac{1}{2}} \|(\Delta_{\mathbb{S}^{d-1}})^{\frac{\alpha}{2}} f\|_{L^2} \\
&\leq C \left(\sum_{n=0}^{\infty} \frac{N(n,d)}{[n(n+d-2)]^{\alpha}} \right)^{\frac{1}{2}} \|f\|_{W^{\alpha,2}}.
\end{aligned}$$

We are left to prove that

$$\sum_{n=0}^{\infty} \frac{N(n,d)}{[n(n+d-2)]^{\alpha}} < \infty. \tag{6.22}$$

By Lemma 6.3.2 and the asymptotics of the Gamma function (iv) in Section 6.1 we have

$$N(n,d) \sim n^{d-2}, \qquad n \to \infty.$$

Hence, since $\alpha > (d-1)/2$, (6.22) holds. $\square$

6.4 Capacity

In this brief section we recall the notion of capacity of a set and its basic properties. We refer to [Lan72] for a complete treatment. Let $s \in (0,d)$ and let $K \subset \mathbb{R}^d$ be compact set. Define

$$I^K_{\mathrm{iso},s}(\mu) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^{-s}\, d\mu(x)\, d\mu(y), \qquad \mu \in \mathcal{P}(K).$$

It is a classical result in potential theory that either $I^K_{\mathrm{iso},s} \equiv +\infty$, or $I^K_{\mathrm{iso},s}$ admits a unique minimizer μ_K. For the existence, one can argue by the Direct Method of the Calculus of Variations. Indeed, any minimizing sequence μ_n admits a subsequence converging narrowly (recall that all the measures μ_n have a common compact support), so that the narrow limit μ is a probability measure on K. Since $I^K_{\mathrm{iso},s}$ is weak-star lower semicontinuous (see, for example, [Lan72, Equation (1.4.5)]), μ is a minimizer. A proof of the uniqueness can be found in [Lan72, pp. 131–133], but can also be deduced as a corollary of Theorem 7.2.1.

We define the s-capacity of K as

$$\mathrm{Cap}_s(K) := \frac{1}{\min\limits_{\mu \in \mathcal{P}(K)} I^K_{\mathrm{iso},s}(\mu)},$$

with the convention that $\mathrm{Cap}_s(K) := 0$ if $I^K_{\mathrm{iso},s} \equiv +\infty$. Note that we are not using the notation of [Lan72], where Cap_s is called $(d-s)$-capacity. For a general set $E \subset \mathbb{R}^d$, one can define the inner and outer s-capacity as follows:

$$\begin{aligned}
\underline{\mathrm{Cap}_s}(E) &:= \sup\left\{\mathrm{Cap}_s(K) : K \subset E \text{ compact}\right\},\\
\overline{\mathrm{Cap}_s}(E) &:= \inf\left\{\underline{\mathrm{Cap}_s}(G) : E \subset G \text{ open}\right\}.
\end{aligned}$$

A set $E \subset \mathbb{R}^d$ is said to be s-capacitable if $\underline{\mathrm{Cap}_s}(E) = \overline{\mathrm{Cap}_s}(E)$, in that case we just write that its s-capacity is $\mathrm{Cap}_s(E)$. One can prove that every Borel set is s-capacitable (see [Lan72, Theorem 2.8]).

6.5 Circulation and curl

We conclude the chapter recalling some simple results regarding the curl of a matrix-valued field. Let $\Omega \subset \mathbb{R}^2$ be a bounded and Lipschitz domain. Given a matrix field $\beta \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^{2\times 2})$, we define the distributional curl of its nth column, for $n = 1, 2$, as the distribution

$$\langle \mathrm{curl}(\beta e_n), \phi \rangle := \int_\Omega \beta e_n \cdot J\nabla\phi \, dx, \qquad \phi \in C^\infty_c(\Omega),$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, one can define the vector-valued distribution $\operatorname{curl}(\beta)$ as the vector whose nth element is $\operatorname{curl}(\beta e_n)$. When the field β is smooth, then Stokes' Theorem gives

$$\int_\Omega \operatorname{curl}(\beta)\, dx = \int_{\partial\Omega} \beta t \, d\mathcal{H}^1,$$

where t is the tangent vector[1] to $\partial\Omega$. The last integral is usually called the circulation of β over $\partial\Omega$. As it is usually done in classical trace theory, one can define a tangent trace for matrix-valued fields in the following space:

$$L^2_{\text{curl}}(\Omega; \mathbb{R}^{2\times 2}) = \{\beta \in L^2(\Omega; \mathbb{R}^{2\times 2}) : \operatorname{curl}(\beta) \in L^2(\Omega; \mathbb{R}^2)\}.$$

Precisely, there exists a unique continuous operator

$$T_t : L^2_{\text{curl}}(\Omega; \mathbb{R}^{2\times 2}) \to W^{-\frac{1}{2},2}(\partial\Omega; \mathbb{R}^2)$$

such that, for

$$\beta \in C^0(\Omega; \mathbb{R}^{2\times 2}) \cap L^2_{\text{curl}}(\Omega; \mathbb{R}^{2\times 2}),$$

it holds $T_t(\beta) = \beta\big|_{\partial\Omega} t$, where t is the tangent vector to $\partial\Omega$ (see, for example, [RJ99, Chapter IX, Theorem 2]). The Stokes' theorem holds in $L^2_{\text{curl}}(\Omega; \mathbb{R}^{2\times 2})$ in the distributional sense, that is

$$\int_\Omega \operatorname{curl}(\beta) \cdot \phi \, dx = \int_\Omega \beta J \nabla \phi \, dx + \langle T_t(\beta), T(\phi) \rangle$$

for every $\phi \in W^{1,2}(\Omega; \mathbb{R}^2)$ and $\beta \in L^2_{\text{curl}}(\Omega; \mathbb{R}^{2\times 2})$, where T is the standard trace operator on $W^{1,2}$.

When $\partial\Omega$ is the union ot two connected components γ_1 and γ_2, then one can identify $W^{-\frac{1}{2},2}(\partial\Omega; \mathbb{R}^2)$ with

$$W^{-\frac{1}{2},2}(\gamma_1; \mathbb{R}^2) + W^{-\frac{1}{2},2}(\gamma_2; \mathbb{R}^2).$$

The Stokes' theorem becomes

$$\int_\Omega \operatorname{curl}(\beta) \cdot \phi \, dx = \int_\Omega \beta J \nabla \phi \, dx + \langle T_t^{\gamma_1}(\beta), T^{\gamma_1}(\phi) \rangle + \langle T_t^{\gamma_2}(\beta), T^{\gamma_2}(\phi) \rangle.$$

It follows that, if $\beta \in L^2_{\text{curl}}(\Omega; \mathbb{R}^{2\times 2})$ has zero curl, then

$$\langle T_t^{\gamma_1}(\beta), T^{\gamma_1}(\phi) \rangle = -\langle T_t^{\gamma_2}(\beta), T^{\gamma_2}(\phi) \rangle,$$

that, in some cases, can be interpreted as the classical principle of path equivalence for conservative fields.

If Ω is simply connected and $\beta \in L^2_{\text{curl}}(\Omega; \mathbb{R}^{2\times 2})$ has zero curl, by a simple approximation argument one can show that $\beta = \nabla P$, for some potential $P \in W^{1,2}(\Omega; \mathbb{R}^2)$. If Ω is not simply connected, then one has $\beta = \nabla P$ only locally in Ω.

[1]The orientation of the tangent vector t is chosen taking the outer normal vector and rotating it counterclockwise by $\pi/2$.

In Chapter 8, we are interested in fields β that are piecewise constant on a finite polyhedral Cacciopoli's partition. Recall that a Cacciopoli's partition is a partition of Ω consisting of sets with finite perimeter. In this case, $\operatorname{curl}(\beta)$ can only concentrate on the straight interfaces between the elements of the partition. In the next simple Lemma, we clarify the cases where no curl concentrates at an interface.

Lemma 6.5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set and let $\{P_i : i = 1, \dots n\}$ be a finite polyhedral Caccioppoli's partiton, that is,*

1. *$\Omega = \cup_{i=1}^n P_i$ and $\operatorname{Per}(P_i) < +\infty$,*
2. *P_i is a polyhedron with boundary given by straight interfaces $S_{1,i}, \dots, S_{n_i,i}$,*
3. *$P_i \cap P_m$ is either empty or is a common interface $S_{j,i} = S_{k,m}$ when $i \neq m$.*

Let $\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be such that $S_{j,i} = S_{\sigma(j,i)}$. Let $\beta : \Omega \to \mathbb{R}^{2\times 2}$ be a matrix-valued field piecewise constant on the partition $\{P_i\}$, that is, $\beta \equiv M_i$ on P_i. Then

$$\operatorname{curl}(\beta) = -\sum_{i=1}^{n} \sum_{j=1}^{n_i} M_i \nu_{j,i} \mathcal{H}^1 \llcorner S_{j,i},$$

where $\nu_{j,i}$ is the tangent vector obtained by rotating counterclokwisely of $\pi/2$ the unit normal to $S_{j,i}$ pointing outside of P_i. In particular, if $(M_i - M_m)\nu_{j,i} = 0$ for some j and m such that $\sigma(j,i) = (k,m)$, then $S_{j,i} = S_{\sigma(j,i)}$ is not contained in the support of $\operatorname{curl} \beta$.

Proof. By an application of Stokes' Theorem, we have that for every $\phi \in C_c^\infty(\Omega; \mathbb{R}^2)$

$$\begin{aligned} \langle \operatorname{curl}(\beta), \phi \rangle &= \sum_{i=1}^{n} \int_{P_i} M_i J \nabla \phi \, dx = -\sum_{i=1}^{n} \int_{\partial P_i} M_i t \cdot \phi \, \mathcal{H}^1 \\ &= -\sum_{i=1}^{n} \sum_{j=1}^{n_i} \int_{S_{j,i}} M_i \nu_{j,i} \cdot \phi \, d\mathcal{H}^1, \end{aligned}$$

concluding the proof. $\square$

7

Nonlocal anisotropic energies with physical confinement

7.1 Assumptions and main results

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$, $d \geq 2$. We consider the repulsive interaction energy

$$I_s(\mu) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{W}_s(x-y)\, d\mu(x) d\mu(y)$$

for $\mu \in \mathcal{P}(\mathbb{R}^d)$, where the interaction kernel $\mathcal{W}_s$, for $s \in (0, d)$, is of the form

$$\mathcal{W}_s(x) := \frac{1}{|x|^s} \Phi\left(\frac{x}{|x|}\right) \tag{7.1}$$

for $x \neq 0$. Here, the profile $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ is a continuous, even, and strictly positive function.

Given a compact set $E \subset \mathbb{R}^d$, we define the confinement potential

$$V_E(x) := \begin{cases} 0 & \text{if } x \in E, \\ +\infty & \text{otherwise.} \end{cases}$$

The attractive-repulsive energy we study is then

$$I_s^E(\mu) := I_s(\mu) + \int_{\mathbb{R}^d} V_E(x)\, d\mu(x)$$

for $\mu \in \mathcal{P}(\mathbb{R}^d)$. Clearly, minimizing I_s^E over $\mathcal{P}(\mathbb{R}^d)$ is equivalent to minimizing I_s over the set of measures $\mu \in \mathcal{P}(\mathbb{R}^d)$ with $\operatorname{supp} \mu \subset E$.

For the choices $E := \overline{B}_1$ and $\Phi \equiv 1$ we denote $I_s^{\overline{B}_1}$ as $I_{\mathrm{iso},s}^{\overline{B}_1}$. The minimizer of $I_{\mathrm{iso},s}^{\overline{B}_1}$ is given by

$$\mu_{\mathrm{iso},s} := \begin{cases} c_{s,d}(1-|x|^2)^{\frac{s-d}{2}}\mathcal{L}^d(x)\llcorner B_1 & \text{if } d-2<s, \\ c_{s,d}\mathcal{H}^{d-1}\llcorner \partial B_1 & \text{if } s \le d-2, \end{cases}$$

where $c_{s,d}$ is a normalization constant (see Lemma 7.3.1 for its exact value).

We focus on super-Coulombic interactions $s \ge d-2$, and confining sets E given by ellipsoids. Our main result is the following.

Theorem 7.1.1. *Let $d \ge 2$, $s \in [d-2, d)$, and let $\mathcal{W}_s$ be as in (7.1) with $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ a continuous, even, and strictly positive function. Assume that $\widehat{\mathcal{W}_s}$ is continuous and non-negative on $\mathbb{R}^d \setminus \{0\}$. Let $E \subset \mathbb{R}^d$ be an ellipsoid of the form $E := RD\overline{B}_1$ for some $R \in \mathrm{SO}(d)$ and some positive-definite $d \times d$ diagonal matrix D, and let T^E be the map $T^E(x) := RDx$ for $x \in \mathbb{R}^d$. Then the unique minimizer of I_s^E over $\mathcal{P}(\mathbb{R}^d)$ is the push-forward of the measure $\mu_{\mathrm{iso},s}$ by the map T^E, that is, the probability measure*

$$\mu_s^E := \begin{cases} |E|^{-1}\dfrac{\Gamma(1+\frac{s}{2})}{\Gamma(1+\frac{d}{2})\Gamma(1+\frac{s-d}{2})}(1-|D^{-1}R^Tx|^2)^{\frac{s-d}{2}}\mathcal{L}^d(x)\llcorner E & \text{if } d-2<s, \\ \left(\mathcal{H}^{d-1}(\partial B_1)\det D\right)^{-1}|D^{-2}R^Tx|^{-1}\mathcal{H}^{d-1}(x)\llcorner \partial E & \text{if } s=d-2. \end{cases}$$

Surprisingly, the minimizing measure μ_s^E is completely independent of the profile Φ, and its support is fully determined by the confinement term. It is natural to ask whether this phenomenon occurs also in the sub-Coulombic regime $s < d-2$. In Section 7.4 we give a negative answer to this question: we consider $E := \overline{B}_1$, and we show that, for a suitable profile Φ, the measure $c_{s,d}\mathcal{H}^{d-1}\llcorner\partial B_1$ does not satisfy the Euler–Lagrange conditions for $I_s^{\overline{B}_1}$ when $s < d-2$.

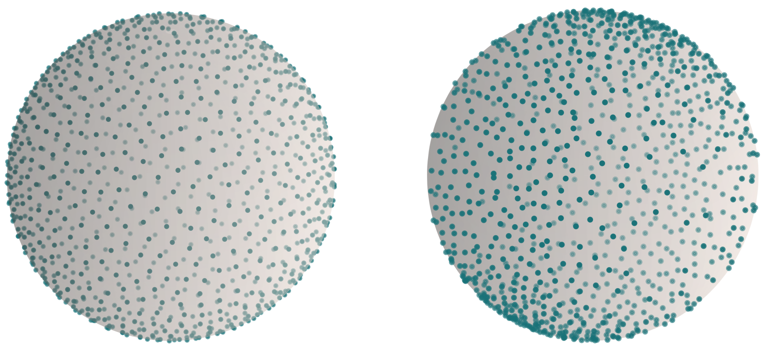

Figure 7.1: Approximated optimal distributions for $d := 3$ and $s := 1/10$, with profiles $\Phi \equiv 1$ (left) and $\Phi := 1 + x_1^2$ (right).

Hence, for $0 < s < d-2$ the anisotropy Φ does play a role in determining the energy minimizers, unlike for $s \ge d-2$ (see Figure 7.1).

7.2 Existence and uniqueness of minimizers

We start the chapter by proving the existence and uniqueness of minimizers and their characterization. This result is by now standard, but for the convenience of the reader we give a self-contained proof.

Theorem 7.2.1. *Let $\mathcal{W}_s$ be a kernel of the form (7.1) with $s \in (0,d)$ and let $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ be a continuous, even, and strictly positive function. Assume that $\widehat{\mathcal{W}_s}$ is continuous and non-negative on $\mathbb{R}^d \setminus \{0\}$. Let $E \subset \mathbb{R}^d$ be a compact set of positive s-capacity. Then the functional I_s^E has a unique minimizer μ over $\mathcal{P}(\mathbb{R}^d)$. Moreover, μ is the unique measure in $\mathcal{P}(\mathbb{R}^d)$ for which there exists a constant $C > 0$ such that*

$$\operatorname{supp}\mu \subset E, \tag{EL1}$$

$$(\mathcal{W}_s * \mu)(x) = C \quad \text{for } \mu\text{-a.e. } x \in \operatorname{supp}\mu, \tag{EL2}$$

$$(\mathcal{W}_s * \mu)(x) \geq C \quad \text{for every } x \in E \setminus N \text{ with } \operatorname{Cap}_s(N) = 0. \tag{EL3}$$

Proof. **Existence.** Since Φ is continuous and strictly positive there exists a constant $C > 0$ such that

$$I_s(\mu) \leq C \int_{\mathbb{R}^d} |x-y|^{-s}\, d\mu(x)\, d\mu(y) \qquad \forall\, \mu \in \mathcal{P}(E).$$

In particular, since $\operatorname{Cap}_s(E) > 0$, $\inf_{\mu \in \mathcal{P}(\mathbb{R}^d)} I_s^E(\mu) < +\infty$.

To prove existence we use the Direct Method of the Calculus of Variations. Let $(\mu_n) \subset \mathcal{P}(\mathbb{R}^d)$ be a minimizing sequence. Without loss of generality we can assume that μ_n has support contained in E for every $n \in \mathbb{N}$. Then, the sequence (μ_n) is tight, and so, up to a subsequence, it converges narrowly to some measure μ. In particular, μ is a probability measure and has support contained in E. Since I_s is lower semicontinuous with respect to the weak-star topology of $\mathcal{M}_b(\mathbb{R}^d)$, we conclude (see, for example, [Lan72, Equation (1.4.5)]).

Uniqueness. We move now to the uniqueness of minimizers. The objective is to show that I_s is strictly convex on the set of probability measures with finite interaction energy and compact support. Since any minimizer belongs to this set, as a by-product we get uniqueness. Firstly, note that $\mathcal{W}_s$ is a tempered distribution, and by Lemma 6.2.7 we can identify $\widehat{\mathcal{W}_s}$ with its continuous restriction to $\mathbb{R}^d \setminus \{0\}$. We show that, given two probability measures $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R}^d)$ with compact support such that $I_s(\mu_1), I_s(\mu_2) < \infty$ and $\mu_1 \neq \mu_2$, it holds

$$\int_{\mathbb{R}^d} \mathcal{W}_s * (\mu_1 - \mu_2)\; d(\mu_1 - \mu_2) > 0. \tag{7.2}$$

Indeed, if this is the case, we immediately deduce strict convexity noting that, for $t \in (0,1)$,

$$\begin{aligned} &I_s(t\mu_1 + (1-t)\mu_2) - tI_s(\mu_1) - (1-t)I_s(\mu_2) \\ &\quad = -t(1-t) \int_{\mathbb{R}^d} \mathcal{W}_s * (\mu_1 - \mu_2)\; d(\mu_1 - \mu_2) < 0. \end{aligned}$$

In order to prove (7.2) we start by showing that, for every finite positive measure $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ with compact support, it holds

$$\int_{\mathbb{R}^d} \mathcal{W}_s * \mu\, d\mu = \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s}(\xi) |\widehat{\mu}(\xi)|^2\, d\xi, \tag{7.3}$$

where the equality is to be understood in the sense that either both sides are finite and coincide or both sides are equal to $+\infty$.

Let (ρ_ε) be a family of standard mollifiers and define $\mu_\varepsilon := \mu * \rho_\varepsilon$. The first step is to prove (7.3) for μ_ε. Note that $\mu_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ so that $\mu_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$. Moreover, since $\mathcal{W}_s$ is a tempered distribution and μ_ε has compact support, the convolution $\mathcal{W}_s * \mu_\varepsilon \in \mathcal{S}'(\mathbb{R}^d)$ and $\widehat{\mathcal{W}_s * \mu_\varepsilon} = \widehat{\mathcal{W}_s}\widehat{\mu}_\varepsilon$ (see also Section 6.2). Since $\widehat{\mathcal{W}_s}$ is $(s-d)$-homogeneous, it behaves as $|\xi|^{s-d}$ at infinity. Then, by the fact that $\widehat{\mu}_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$, we have $\widehat{\mathcal{W}_s}\widehat{\mu}_\varepsilon \in L^1(\mathbb{R}^d)$. Thus, we are in a position to apply Parseval's Formula (see Lemma 6.2.3 and Remark 6.2.4) and deduce that

$$\begin{aligned}\int_{\mathbb{R}^d} (\mathcal{W}_s * \mu_\varepsilon)(x)\mu_\varepsilon(x)\,dx &= \int_{\mathbb{R}^d} \overline{\widehat{\mathcal{W}_s}(\xi)\widehat{\mu}_\varepsilon(\xi)}\widehat{\mu}_\varepsilon(\xi)\,d\xi \\ &= \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s}(\xi)|\widehat{\mu}_\varepsilon(\xi)|^2\,d\xi,\end{aligned} \tag{7.4}$$

where we have used the fact that $\widehat{\mathcal{W}_s}$ is real by parity. This concludes the proof of (7.3) for μ_ε.

We wish now to pass to the limit as $\varepsilon \to 0$. Firstly, note that $\widehat{\mu}_\varepsilon = \widehat{\mu}\widehat{\rho}_\varepsilon$. Indeed, μ has compact support and ρ_ε is a Schwarz function. Moreover, $\widehat{\rho}_\varepsilon(x) = \widehat{\rho}_1(\varepsilon x)$, by definition of standard mollifier. Thus, $\widehat{\mu}_\varepsilon \to \widehat{\mu}$ pointwise. Note also that $|\widehat{\rho}_\varepsilon(\xi)| \le 1$. We distinguish two cases: if $\widehat{\mathcal{W}_s}|\widehat{\mu}|^2 \in L^1(\mathbb{R}^d)$, then we can pass to the limit in the right-hand side of (7.4) by Dominated Convergence Theorem; otherwise, we apply Fatou's Lemma to deduce that the limit is $+\infty$ (recall that $\widehat{\mathcal{W}_s} \ge 0$). In any case, we have

$$\int_{\mathbb{R}^d} \widehat{\mathcal{W}_s}(\xi)|\widehat{\mu}_\varepsilon(\xi)|^2\,d\xi \to \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s}(\xi)|\widehat{\mu}(\xi)|^2\,d\xi.$$

For the left-hand side, we note that

$$\int_{\mathbb{R}^d} (\mathcal{W}_s * \mu_\varepsilon)(x)\mu_\varepsilon(x)\,dx = \int_{\mathbb{R}^d} (\mathcal{W}_s * \rho_\varepsilon * \rho_\varepsilon)(x-y)\,d\mu(x)\,d\mu(y). \tag{7.5}$$

Let $\eta_\varepsilon := \rho_\varepsilon * \rho_\varepsilon$. It is immediate to show that $\eta_\varepsilon(x) = \varepsilon^{-d}(\rho * \rho)(x/\varepsilon)$ and that $\rho * \rho$ satisfies all the properties of a standard mollifier: it is radial, smooth with compact support, bounded by 1, and has integral equal to 1. Thus, without loss of generality, we can suppose that (η_ε) is a family of standard mollifiers. In particular, since $\mathcal{W}_s$ is continuous outside the origin, $\mathcal{W}_s * \eta_\varepsilon \to \mathcal{W}_s$ everywhere in $\mathbb{R}^d \setminus \{0\}$. Let us denote by $\mathcal{W}_s^{\mathrm{iso}}$ the isotropic Riesz Kernel $\mathcal{W}_s^{\mathrm{iso}}(x) := |x|^{-s}$. Since the anisotropy profile Φ is continuous on $\mathbb{S}^{d-1}$ and strictly positive, there is a constant $C > 0$ such that

$$\frac{1}{C}\mathcal{W}_s^{\mathrm{iso}}(x) \le \mathcal{W}_s(x) \le C\mathcal{W}_s^{\mathrm{iso}}(x). \tag{7.6}$$

In particular, for $\varepsilon < 1$ and $x \neq 0$, we have

$$\begin{aligned}(\mathcal{W}_s * \eta_\varepsilon)(x) &= \int_{B_\varepsilon(x)} \mathcal{W}_s(y)\eta_\varepsilon(x-y)\,dy \le C\int_{B_1(0)} \mathcal{W}_s^{\mathrm{iso}}(\varepsilon y - x)\,dy \\ &\le C\varepsilon^{-s}\int_{B_1(0)} \mathcal{W}_s^{\mathrm{iso}}(y - x/\varepsilon)\,dy = C\mathcal{W}_s^{\mathrm{iso}}(x)\int_{B_1(0)} \frac{|x/\varepsilon|^s}{|y-x/\varepsilon|^s}\,dy \\ &\le C^2\mathcal{W}_s(x)\int_{B_1(0)} \frac{|x/\varepsilon|^s}{|y-x/\varepsilon|^s}\,dy =: C\mathcal{W}_s(x)f(x/\varepsilon).\end{aligned}$$

We show that f is bounded, which gives that $\mathcal{W}_s * \eta_\varepsilon$ is dominated by $\mathcal{W}_s$. Indeed, for $|x| > M$, with $M > 0$ sufficiently large, we have

$$f(x) = \int_{B_1(0)} \frac{|x|^s}{|y-x|^s}\, dy \le \int_{B_1(x)} \frac{|x|^s}{|y|^s}\, dy \le \int_{B_1(x)} \frac{|x|^s}{(|x|-1)^s}\, dy \le C.$$

Instead, for $|x| \le M$

$$f(x) \le M^s \int_{B_1(0)} \frac{1}{|y-x|^s}\, dy = M^s \int_{B_1(x)} \frac{1}{|y^s|}\, dy \le M^s \int_{B_{M+1}(0)} \frac{1}{|y|^s}\, dy \le C.$$

Thus, the supremum of f is finite and so is bounded. As before, we distinguish two cases: if $(x,y) \mapsto \mathcal{W}_s(x-y)$ is integrable with respect to the product measure $\mu \otimes \mu$, then we apply Dominated Convergence Theorem; otherwise, we resort to Fatou's Lemma. In any case, passing to the limit in the left-hand side of (7.4) (recall also (7.5)) we get

$$\int_{\mathbb{R}^d} (\mathcal{W}_s * \mu_\varepsilon)(x)\mu_\varepsilon(x)\, dx \to \int_{\mathbb{R}^d} (\mathcal{W}_s * \mu)\, d\mu.$$

In particular, we have proved (7.3) for finite positive measures μ. We show now that (7.3) holds also for a signed measure of the form $\mu_1 - \mu_2$, with μ_1, μ_2 being two finite and positive measures with finite interaction energy. First, observe that by (7.3)

$$\int_{\mathbb{R}^d} \mathcal{W}_s * (\mu_1+\mu_2)\, d(\mu_1+\mu_2) = \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s} |\widehat{\mu}_1 + \widehat{\mu}_2|^2\, d\xi. \tag{7.7}$$

Expanding both sides of (7.7), recalling that both μ_1 and μ_2 have finite interaction energy, and applying once again (7.3), we get

$$\int_{\mathbb{R}^d} \mathcal{W}_s * \mu_1\, d\mu_2 = \int_{\mathbb{R}^d} \mathcal{W}_s * \mu_2\, d\mu_1 = \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s} \Re(\widehat{\mu}_1 \overline{\widehat{\mu}_2})\, d\xi. \tag{7.8}$$

Then, by (7.8), and a further application of (7.3), we deduce

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{W}_s * (\mu_1-\mu_2)\, d(\mu_1-\mu_2) &= \int_{\mathbb{R}^d} \mathcal{W}_s * \mu_1\, d\mu_1 + \int_{\mathbb{R}^d} \mathcal{W}_s * \mu_2\, d\mu_2 \\ &\quad - 2\int_{\mathbb{R}^d} \mathcal{W}_s * \mu_1\, d\mu_2 = \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s} |\widehat{\mu}_1|^2\, d\xi \\ &\quad + \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s} |\widehat{\mu}_2|^2\, d\xi - 2\int_{\mathbb{R}^d} \widehat{\mathcal{W}_s} \Re(\widehat{\mu}_1 \overline{\widehat{\mu}_2})\, d\xi \\ &= \int_{\mathbb{R}^d} \widehat{\mathcal{W}_s} |\widehat{\mu}_1 - \widehat{\mu}_2|^2\, d\xi.\end{aligned} \tag{7.9}$$

We are left to prove (7.2). If $\widehat{\mathcal{W}_s} > 0$, then by (7.9) we are done. However, some care is needed if $\widehat{\mathcal{W}_s} \ge 0$. Note that we have $\widehat{\mathcal{W}_s} \ne 0$, since $\mathcal{W}_s \ne 0$. In this case, we show that for any measure μ with compact support the set of zeros

$$\mathcal{Z}(\widehat{\mu}) := \{\xi \in \mathbb{R}^d : \widehat{\mu}(\xi) = 0\}$$

has zero Lebesgue measure. Then, (7.2) follows immediately by (7.9). It is well-known that the zero set of any analytic function has zero Lebesgue measure. For a simple proof of this fact, we refer to [Mit20]. As we already recalled in Section 6.2, the Fourier's transform of a compactly supported distribution can be extended to the whole $\mathbb{C}^d$, and this extension is entire. This in particular implies that its restriction to $\mathbb{R}^d$ is analytic.

Euler–Lagrange equations. We start by showing that the minimizer μ satisfies the Euler–Lagrange equations. Note that, since $\min I_s^E < +\infty$, μ has support contained in E, in other words (EL1) holds and the support of μ is compact. Consider the variations $(1-\varepsilon)\mu + \varepsilon\nu$, where $\nu \in \mathcal{P}(\mathbb{R}^d)$ has support contained in E and finite interaction energy $I_s(\nu) < +\infty$. Then, by minimality, for every $\varepsilon \in (0,1)$

$$I_s(\mu) \le I_s((1-\varepsilon)\mu + \varepsilon\nu),$$

from which we deduce

$$\varepsilon^2 \int_{\mathbb{R}^d} \mathcal{W}_s * (\mu - \nu)\, d(\mu - \nu) + 2\varepsilon \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\nu - 2\varepsilon \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \; d\mu \ge 0. \tag{7.10}$$

Note that we have used the following rewriting of the energy

$$I_s(\mu) = \int_{\mathbb{R}^d} \mathcal{W}_s(x-y)\, d\mu(x)\, d\mu(y) = \int_{\mathbb{R}^d} (\mathcal{W}_s * \mu)(x)\, d\mu(x).$$

If

$$\int_{\mathbb{R}^d} \mathcal{W}_s * (\mu - \nu)\, d(\mu - \nu),\ \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\nu < +\infty, \tag{7.11}$$

we divide (7.10) by ε and we pass to the limit as $\varepsilon \to 0$, obtaining

$$\int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\nu \ge \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \; d\mu =: C. \tag{7.12}$$

Note that (7.12) holds regardless of the validity of (7.11), since μ has finite interaction energy. We show now that (7.12) implies (EL3). Suppose by contradiction that the set

$$A := \{x \in \mathbb{R}^d : (\mathcal{W}_s * \mu)(x) < C\} \cap E$$

has positive s-capacity. Note that it is s-capacitable since it is a Borel set. Then, by definition of capacity (see Section 6.4), for $n \gg 1$ the set

$$K := \left\{x \in \mathbb{R}^d : (\mathcal{W}_s * \mu)(x) \le C - \frac{1}{n}\right\} \cap E \tag{7.13}$$

has positive s-capacity, and by lower semicontinuity of $\mathcal{W}_s * \mu$ (see [Lan72, Lemma 0.1]) it is compact. By definition of C in (7.12), there exists a Borel set B with positive measure, disjoint from K, such that

$$(\mathcal{W}_s * \mu)(x) > C - \frac{1}{2n}, \qquad \mu\text{-a.e. in } B.$$

By definition of s-capacity of a compact set, there exists a measure $\bar{\nu} \in \mathcal{P}(K)$ such that

$$\mathrm{Cap}_s(K)^{-1} = \int_{\mathbb{R}^d} |x-y|^{-s}\, d\bar{\nu}(x)\, d\bar{\nu}(y) < +\infty.$$

Define, for $\varepsilon \ll 1$, the probability measure

$$\nu := \mu + \varepsilon\mu(B)\bar{\nu} - \varepsilon\mu \llcorner B.$$

Then, by (7.12)

$$\begin{aligned} C &= \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\mu \le \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\nu = C + \varepsilon\mu(B) \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\bar{\nu} - \varepsilon \int_B \mathcal{W}_s * \mu \, d\mu \\ &\le C + \varepsilon\mu(B)\Big(C - \frac{1}{n}\Big) - \varepsilon\mu(B)\Big(C - \frac{1}{2n}\Big) \le C - \frac{\varepsilon\mu(B)}{2n}, \end{aligned}$$

giving a contradiction.

We show now that (EL2) is a consequence of (EL3) and the definition of C. Firstly, observe that (EL3) holds μ-a.e.. Indeed, suppose by contradiction that the set A is such that $\mu(A) > 0$, while being of zero s-capacity. Then, for $n \gg 1$, the compact set K as defined in (7.13) has zero capacity and satisfies $\mu(K) > 0$. By (7.6), since $I_s^E(\mu) < +\infty$, we immediately deduce that $I_{\mathrm{iso},s}(\mu) < +\infty$. Thus, the probability measure $\nu := (\mu(K))^{-1}\mu \llcorner K$ has finite interaction energy $I_{\mathrm{iso},s}(\nu) < +\infty$, contradicting the zero capacity of K. Therefore, $\mu(A) = 0$. By definition of C,

$$C = \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\mu = C\mu(\{x \in \mathbb{R}^d : (\mathcal{W}_s * \mu)(x) = C\}) + \int_{\{\mathcal{W}_s * \mu > C\}} \mathcal{W}_s * \mu \, d\mu.$$

If

$$\mu(\{x \in E : (\mathcal{W}_s * \mu)(x) > C\}) > 0,$$

we have a contradiction.

To conclude the proof we show that (EL1)–(EL3) imply minimality. Let $\bar{\mu}, \mu \in \mathcal{P}(\mathbb{R}^d)$ satisfy (EL1)–(EL3) for some constants $\bar{C}, C > 0$, respectively. Integrating (EL2) we deduce that

$$\begin{aligned} \bar{C} &= \int_{\mathbb{R}^d} \mathcal{W}_s * \bar{\mu} \, d\bar{\mu}, \\ C &= \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\mu. \end{aligned}$$

For $\varepsilon \in (0,1)$, define $\mu_\varepsilon := \varepsilon\mu + (1-\varepsilon)\bar{\mu}$. Arguing as before, we can show that (EL3) holds also μ-a.e. (resp. $\bar{\mu}$-a.e.). Thus, we have

$$\begin{aligned} I_s^E(\mu_\varepsilon) &= \varepsilon \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\mu_\varepsilon + (1-\varepsilon) \int_{\mathbb{R}^d} \mathcal{W}_s * \bar{\mu} \, d\mu_\varepsilon \ge \varepsilon C + (1-\varepsilon)\bar{C} \\ &= \varepsilon \int_{\mathbb{R}^d} \mathcal{W}_s * \mu \, d\mu + (1-\varepsilon) \int_{\mathbb{R}^d} \mathcal{W}_s * \bar{\mu} \, d\bar{\mu} = \varepsilon I_s(\mu) + (1-\varepsilon) I_s(\bar{\mu}). \end{aligned}$$

Since $I_s^E(\mu_\varepsilon) = I_s(\mu_\varepsilon)$ and I_s is strictly convex (see the proof of uniqueness), we conclude that $\mu = \bar{\mu}$, showing that the only measure satisfying the Euler–Lagrange equations is the minimizer. □

7.3 Characterization of the minimizer in the super-Coulombic case

In this section we prove Theorem 7.1.1. A general ellipsoid in $\mathbb{R}^d$ centred at the origin can be described as $E := RD\overline{B}_1$, where $R \in \mathrm{SO}(d)$, D is a positive-definite

$d \times d$ diagonal matrix, and $\overline{B}_1$ is the closed unit ball centred at the origin. Given an ellipsoid $E := RD\overline{B}_1$, we define the linear map $T^E(x) := RDx$ for $x \in \mathbb{R}^d$.

For any $q \geq d-2$, we define

$$\mu_q := \begin{cases} c_{q,d}(1-|x|^2)^{\frac{q-d}{2}} \mathcal{L}^d(x) \llcorner B_1 & \text{if } q > d-2, \\ c_{q,d} \mathcal{H}^{d-1} \llcorner \partial B_1 & \text{if } q = d-2, \end{cases} \tag{7.14}$$

where $c_{q,d}$ is a normalization constant so that μ_q is a probability measure. For completeness, we compute it in the next Lemma.

Lemma 7.3.1. *We have*

$$c_{q,d} = \begin{cases} |B_1|^{-1} \dfrac{\Gamma(1+\frac{q}{2})}{\Gamma(1+\frac{d}{2})\Gamma(1+\frac{q-d}{2})} = \pi^{-\frac{d}{2}} \dfrac{\Gamma(1+\frac{q}{2})}{\Gamma(1+\frac{q-d}{2})} & \textit{if } q > d-2, \\ \left(\mathcal{H}^{d-1}(\partial B_1)\right)^{-1} = \pi^{-\frac{d}{2}} \dfrac{\Gamma(\frac{d}{2})}{2} & \textit{if } q = d-2. \end{cases}$$

Proof. For $q = d-2$, there is nothing to prove, so we focus on the case $q > d-2$. By Formula 6.1.5 with $\mu = d$, $\nu = (q-d+2)/2$, and $\lambda = 2$, we have

$$\begin{aligned} \int_{B_1} (1-|x|^2)^{\frac{q-d}{2}} \, dx &= \int_{\partial B_1} \int_0^1 \rho^{d-1}(1-\rho^2)^{\frac{q-d}{2}} \, d\rho \, d\omega \\ &= \frac{1}{2} \mathcal{H}^{d-1}(\partial B_1) B\left(\frac{d}{2}, \frac{q-d+2}{2}\right) = \pi^{\frac{d}{2}} \frac{\Gamma(1+\frac{q-d}{2})}{\Gamma(1+\frac{q}{2})}, \end{aligned}$$

where we have used that $\mathcal{H}^{d-1}(\partial B_1) = 2\pi^{\frac{d}{2}}/\Gamma(d/2)$. $\square$

For $s = q \in [d-2, d) \cap (0,d)$, μ_q clearly coincides with $\mu_{\mathrm{iso},s}$. The push-forward μ_q^E of μ_q by the map T^E is given by

$$\mu_q^E = \begin{cases} |E|^{-1} \dfrac{\Gamma(1+\frac{q}{2})}{\Gamma(1+\frac{d}{2})\Gamma(1+\frac{q-d}{2})} (1-|D^{-1}R^T x|^2)^{\frac{q-d}{2}} \mathcal{L}^d(x) \llcorner E & \text{if } q > d-2, \\ \left(\mathcal{H}^{d-1}(\partial B_1) \det D\right)^{-1} |D^{-2}R^T x|^{-1} \mathcal{H}^{d-1}(x) \llcorner \partial E & \text{if } q = d-2. \end{cases}$$

Before moving to the proof of our main result, it might be worth pointing out that asking $\widehat{\mathcal{W}_s}$ to be continous on $\mathbb{R}^d \setminus \{0\}$ is not restrictive, and can be obtained, for example, by asking enough regularity of the anisotropic profile Φ.

Proposition 7.3.2. *Let $d \geq 2$, $s \in (0,d)$, and let $\mathcal{W}_s$ be as in (7.1) with $\Phi \in W^{\alpha,2}(\mathbb{S}^{d-1})$ an even, and strictly positive function. Then, if $\alpha > d-s-1/2$, Φ is continuous on $\mathbb{S}^{d-1}$ and $\widehat{\mathcal{W}_s}$ is continuous on $\mathbb{R}^d \setminus \{0\}$.*

Proof. Note that by the embedding proved in Theorem 6.3.3, Φ is continuous on $\mathbb{S}^{d-1}$. To compute the Fourier's transform of $\mathcal{W}_s$, it is convenient to write the profile $\Phi \in L^2(\mathbb{S}^{d-1})$ in terms of spherical harmonics. Let

$$\left\{Y_n^i : n \in \mathbb{N}, i = 1, \ldots, N(n,d)\right\} \subset L^2(\mathbb{S}^{d-1})$$

be an orthonormal basis of $L^2(\mathbb{S}^{d-1})$ whose elements are spherical harmonics. Here, $N(n,d)$ is defined as in Lemma 6.3.2. Recall that Y_n^i is the restriction to $\mathbb{S}^{d-1}$ of an harmonic homogenous polynomial of degree n, that, with a small abuse of notation, we still denote by Y_n^i. We can write

$$\Phi = \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} \lambda_n^i Y_n^i, \qquad \lambda_n^i \in \mathbb{R}.$$

Then for $x \in \mathbb{R}^d$

$$\mathcal{W}_s(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} \frac{\lambda_n^i}{|x|^s} Y_n^i \left(\frac{x}{|x|} \right) = \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} \frac{\lambda_n^i}{|x|^{s+n}} Y_n^i(x).$$

By Lemma 6.2.8 we infer that for $\xi \in \mathbb{R}^d$

$$\widehat{\mathcal{W}_s}(\xi) = \frac{1}{|\xi|^{d-s}} \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} \lambda_n^i (-1)^n 2^{d-s} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{n+d-s}{2})}{\Gamma(\frac{n+s}{2})} Y_n^i \left(\frac{\xi}{|\xi|} \right),$$

provided the series at the right-hand side converges in $L^2(\mathbb{S}^{d-1})$. We show that

$$\sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} (\lambda_n^i)^2 \frac{\Gamma^2(\frac{n+d-s}{2})}{\Gamma^2(\frac{n+s}{2})} < \infty,$$

that gives the convergence in $L^2(\mathbb{S}^{d-1})$. By the asymptotic behaviour of the Gamma function (iv) in Section 6.1, we have

$$\frac{\Gamma^2(\frac{n+d-s}{2})}{\Gamma^2(\frac{n+s}{2})} \sim n^{d-2s}, \tag{7.15}$$

Since $\alpha > d - s - 1/2$, by the definition of $W^{\alpha,2}(\mathbb{S}^{d-1})$ we have the desired convergence. Let

$$\Psi := \sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} \lambda_n^i (-1)^n 2^{d-s} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{n+d-s}{2})}{\Gamma(\frac{n+s}{2})} Y_n^i \left(\frac{\xi}{|\xi|} \right) \in L^2(\mathbb{S}^{d-1}).$$

To conclude we show that $\Psi \in W^{\beta,2}(\mathbb{S}^{d-1})$, for some $\beta > (d-1)/2$. The continuity of $\widehat{\mathcal{W}_s}$ then follows from Theorem 6.3.3. By definition of $W^{\beta,2}(\mathbb{S}^{d-1})$ (see Section 6.3), we need to prove that

$$\sum_{n=0}^{\infty} \sum_{i=1}^{N(n,d)} (\lambda_n^i)^2 (n(n+d-2))^{\beta} \frac{\Gamma^2(\frac{n+d-s}{2})}{\Gamma^2(\frac{n+s}{2})} < \infty.$$

By (7.15) we have

$$(n(n+d-2))^{\beta} \frac{\Gamma^2(\frac{n+d-s}{2})}{\Gamma^2(\frac{n+s}{2})} \sim n^{d-2s+2\beta}.$$

Since $\alpha > d - s - 1/2$, we conclude. $\square$

We move to the proof of the Fourier's transform of μ_q and μ_q^E. Note that μ_q and μ_q^E are Radon measures with compact support, hence they are tempered distributions.

Lemma 7.3.3. *Let $q \geq d-2$, and let $\mu_q \in \mathcal{P}(\mathbb{R}^d)$ be defined as in (7.14). Then*

$$\widehat{\mu}_q(\xi) = \hat{c}_q \frac{1}{|\xi|^{\frac{q}{2}}} J_{\frac{q}{2}}(|\xi|),$$

where

$$\hat{c}_q := 2^{\frac{q}{2}} \Gamma\left(1 + \frac{q}{2}\right).$$

Moreover, if E is an ellipsoid of the form $E = RD\overline{B}_1$ with $R \in \mathrm{SO}(d)$ and D a positive-definite $d \times d$ diagonal matrix, then

$$\widehat{\mu}_q^E(\xi) = \widehat{\mu}_q(DR^T\xi). \tag{7.16}$$

Proof. We start with the case $q > d-2$. Applying Formula 6.2.5 for the Fourier's transform of a radial function we get

$$\widehat{\mu}_q(\xi) = c_{q,d} \frac{(2\pi)^{\frac{d}{2}}}{|\xi|^{\frac{d}{2}-1}} \int_0^1 (1-r^2)^{\frac{q-d}{2}} J_{\frac{d}{2}-1}(r|\xi|) r^{\frac{d}{2}}\, dr. \tag{7.17}$$

To compute the integral in (7.17) we use Formula 6.1.3 with $\nu = d/2 - 1$, $\rho = (q-d)/2$, and $b = |\xi|$. We obtain

$$\widehat{\mu}_q(\xi) = c_{q,d} \pi^{\frac{d}{2}} 2^{\frac{q}{2}} \Gamma\left(1 + \frac{q-d}{2}\right) \frac{1}{|\xi|^{\frac{q}{2}}} J_{\frac{q}{2}}(|\xi|) = 2^{\frac{q}{2}} \Gamma\left(1 + \frac{q}{2}\right) \frac{1}{|\xi|^{\frac{q}{2}}} J_{\frac{q}{2}}(|\xi|),$$

which proves the first claim in the statement. Here, we have used the explicit expression of $c_{q,d}$ given in Lemma 7.3.1. For a similar computation see also [Gra08, Appendix B.5], where a slightly different definition of the Fourier's transform is used.

When $q = d-2$, we use (6.15) to get

$$\widehat{\mu}_{d-2}(\xi) = 2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right) \frac{1}{|\xi|^{\frac{d}{2}-1}} J_{\frac{d}{2}-1}(|\xi|).$$

Finally, (7.16) follows from the linearity and invertibility of the map T^E. □

The key ingredient for the proof of our main result, Theorem 7.1.1, is a formula for the expression of the potential function $\mathcal{W}_s * \mu_q^E$ inside E when $s \in (0,d)$ and $q \in [d-2, +\infty)$. As a first step, in the next technical lemma we study the regularity of $\mathcal{W}_s * \mu_q^E$.

Lemma 7.3.4. *Suppose $s \in (0,d)$ and $q \in [d-2, +\infty)$. Let $\mathcal{W}_s$ be as in (7.1) with $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ a continuous, even, and strictly positive function. Let $E := RD\overline{B}_1$ be an ellipsoid, and let μ_q^E be the push-forward of the measure μ_q by the map T^E. Then*

$$\mathcal{W}_s * \mu_q^E \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap C^0(\mathbb{R}^d \setminus \partial E).$$

Moreover, if $0 < s < \min(d, (q+d)/2)$, then $\mathcal{W}_s * \mu_q^E \in C^0(\mathbb{R}^d)$.

Proof. We only sketch the proof in the case $E := \overline{B}_1$ and $\Phi \equiv 1$, and for convenience we ignore the normalization constant $c_{q,d}$. It is immediate to show that, for any $0 < s < d$ and $q > d-2$, we have

$$\mathcal{W}_s * \mu_q \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap C^0(\mathbb{R}^d \setminus \partial B_1).$$

The continuity in $\mathbb{R}^d \setminus \partial B_1$ also holds for $q = d-2$. We focus on the behaviour of the potential across ∂B_1. To this aim we consider, with no loss of generality, $(\mathcal{W}_s * \mu_q)(te_1)$ for t close to 1. We use the compact notation

$$(\mathcal{W}_s * \mu_q)(te_1) = \begin{cases} v_s(t,1) & \text{if } q = d-2, \\ \displaystyle\int_0^1 v_s(t,r)(1-r^2)^{\frac{q-d}{2}}\, dr & \text{if } q > d-2, \end{cases} \tag{7.18}$$

where

$$v_s(t,r) := \int_{\partial B_r} \frac{1}{|te_1 - y|^s}\, d\mathcal{H}^{d-1}(y),$$

for $t, r > 0$ and $t \neq r$. By setting $y := r\omega$ and using spherical coordinates on ∂B_1 we have

$$\begin{aligned} v_s(t,r) &= C(d) r^{d-1} \int_0^\pi \frac{\sin^{d-2}(\varphi)}{(t^2 + r^2 - 2rt\cos(\varphi))^{s/2}}\, d\varphi \\ &= \frac{C(d) r^{d-1}}{h^s} \int_0^\pi \frac{\sin^{d-2}(\varphi)}{(1 + \alpha^2 - 2\alpha\cos(\varphi))^{s/2}}\, d\varphi, \end{aligned}$$

where $h := \max(t,r)$, $\alpha := \min(t,r)/h \in (0,1)$, and $C(d) > 0$ is a dimensional constant. The integral above can be computed explicitly using Formula 6.1.2 with

$$\mu = \frac{(d-1)}{2}, \quad \nu = \frac{s}{2}, \quad a = \alpha,$$

and gives

$$v_s(t,r) = \frac{C(d) r^{d-1}}{h^s} B\Big(\frac{d-1}{2}, \frac{1}{2}\Big)\, {}_2F_1\Big(\frac{s}{2}, \frac{s-d+2}{2}; \frac{d}{2}; \alpha^2\Big). \tag{7.19}$$

We now treat the cases $0 < s < d-1$ and $d-1 \leq s < d$ separately.

Let $0 < s < d-1$. As recalled in Section 6.1, ${}_2F_1$ is continuous with respect to the last variable in the interval $[0,1]$ whenever

$$\frac{d}{2} - \frac{s}{2} - \frac{s-d+2}{2} > 0,$$

namely for $0 < s < d-1$. Hence, by (7.19), v_s is continuous in t. For $q = d-2$, this immediately provides the continuity of the potential in the whole $\mathbb{R}^d$. For $q > d-2$ the continuity follows by (7.18), (7.19), and the Dominated Convergence Theorem. Note that for $0 < s < d-1$ and $q \geq d-2$, we have

$$\min\left(d, \frac{q+d}{2}\right) \geq d-1 > s.$$

Let now $d-1<s<d$. In this range of Riesz exponents, by (6.3),

$$ {}_2F_1\left(\frac{s}{2}, \frac{s-d+2}{2}; \frac{d}{2}; \alpha^2\right) \sim (1-\alpha^2)^{d-s-1} \qquad \text{for } \alpha \sim 1. \tag{7.20} $$

If $q=d-2$, the asymptotics in (7.20), with (7.18)–(7.19) and the fact that $d-s-1>-1$, implies that $\mathcal{W}_s * \mu_q$ is integrable in a neighbourhood of ∂B_1, hence it is locally integrable in $\mathbb{R}^d$. The same asymptotics shows that the potential blows up on ∂B_1, thus it is not continuous on $\mathbb{R}^d$. Note that for $d-1<s<d$ and $q=d-2$, we have

$$ \min\left(d, \frac{q+d}{2}\right) = d-1 < s. $$

Let now $q>d-2$. Let $0<\varepsilon<1/4$, and assume that $|t-1|<\varepsilon$. In what follows we consider the integral

$$ \int_{1-2\varepsilon}^{1} v_s(t,r)(1-r^2)^{\frac{q-d}{2}}\, dr, \tag{7.21} $$

which is crucial in proving the continuity of the potential. Since $|t-1|<\varepsilon$, $1-2\varepsilon<r<1$, and $t\neq r$, we have

$$ \frac{1-2\varepsilon}{1+\varepsilon} \leq \alpha < 1. $$

Hence, for ε small enough, we can replace, up to constants, the hypergeometric function in (7.19) with its asymptotics given by (7.20). Namely, we can estimate (7.21) as follows:

$$ \int_{1-2\varepsilon}^{1} v_s(t,r)(1-r^2)^{\frac{q-d}{2}}\, dr \sim \int_{1-2\varepsilon}^{1} \frac{r^{d-1}}{h^s}(1-\alpha^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr =: I. $$

We distinguish two cases, depending on whether t is smaller or greater than 1.

Let $t<1$. We split I into $I=I_1+I_2$, where

$$ \begin{aligned} I_1 &:= \int_{1-2\varepsilon}^{t} \frac{r^{d-1}}{h^s}(1-\alpha^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr \\ &= \int_{1-2\varepsilon}^{t} \frac{r^{d-1}}{t^{2d-s-2}}(t^2-r^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr, \\ I_2 &:= \int_{t}^{1} \frac{r^{d-1}}{h^s}(1-\alpha^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr \\ &= \int_{t}^{1} r^{s+1-d}(r^2-t^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr. \end{aligned} $$

To estimate I_2 it is convenient to further split $I_2 = I_3 + I_4$, where

$$ \begin{aligned} I_3 &:= \int_{t}^{\frac{1+t}{2}} r^{s+1-d}(r^2-t^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr, \\ I_4 &:= \int_{\frac{1+t}{2}}^{1} r^{s+1-d}(r^2-t^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr. \end{aligned} $$

For $q < d$ we have

$$|I_3| \le (2t)^{d-s-1}(1+t)^{\frac{q-d}{2}} \left(1 - \frac{1+t}{2}\right)^{\frac{q-d}{2}} \int_t^{\frac{1+t}{2}} (r-t)^{d-s-1}\, dr$$
$$\le C_1 \left(1 - \frac{1+t}{2}\right)^{\frac{q-d}{2}} \left(\frac{1+t}{2} - t\right)^{d-s} = C_1 \left(\frac{1-t}{2}\right)^{\frac{q+d}{2}-s} = C_2\, (1-t)^{\frac{q+d}{2}-s},$$

where $C_1, C_2 > 0$, and we used that $t \le r \le (1+t)/2$ and $s+1-d > 0$. An analogous reasoning leads to the same estimate for $q \ge d$. Similarly, since $q > d-2$,

$$|I_4| \le C_3(1-t)^{d-s-1} \int_{\frac{1+t}{2}}^1 (1-r)^{\frac{q-d}{2}}\, dr = C_3(1-t)^{d-s-1}\left(1 - \frac{1+t}{2}\right)^{\frac{q-d+2}{2}}$$
$$= C_4(1-t)^{\frac{q+d}{2}-s},$$

where $C_3, C_4 > 0$, and we used that $r - t \ge (1-t)/2$. As for the term I_1, setting $\tau := t - r$, we obtain

$$I_1 = \int_0^{t-1+2\varepsilon} \frac{(t-\tau)^{d-1}}{t^{2d-s-2}} \tau^{d-s-1}(2t-\tau)^{d-s-1}(1-(t-\tau)^2)^{\frac{q-d}{2}}\, d\tau$$
$$\le C_5 \int_0^{t-1+2\varepsilon} \tau^{d-s-1}(1-t+\tau)^{\frac{q-d}{2}}\, d\tau \le C_6 \int_0^{t-1+2\varepsilon} f(t,\tau)\, d\tau,$$

where $C_5, C_6 > 0$, and

$$f(t,\tau) := \begin{cases} \tau^{d-s-1} & \text{if } \frac{q-d}{2} \ge 0, \\ \tau^{\frac{q+d}{2}-s-1} & \text{if } -1 < \frac{q-d}{2} < 0. \end{cases}$$

Hence, we deduce the estimate

$$|I_1| \le C_7 \begin{cases} (t-1+2\varepsilon)^{d-s} & \text{if } \frac{q-d}{2} \ge 0, \\ (t-1+2\varepsilon)^{\frac{q+d}{2}-s} & \text{if } -1 < \frac{q-d}{2} < 0 \end{cases}$$

for some $C_7 > 0$. In both cases, if $0 < s < \min(d, (q+d)/2)$, we can find a positive β such that

$$|I_1| \le C_7(t-1+2\varepsilon)^\beta \le C_7(2\varepsilon)^\beta.$$

For $t > 1$, instead, we write

$$I = \int_{1-2\varepsilon}^1 \frac{r^{d-1}}{t^{2d-s-2}}(t^2-r^2)^{d-s-1}(1-r^2)^{\frac{q-d}{2}}\, dr,$$

and, assuming $(q+d)/2 - s > 0$, we obtain

$$|I| \le C_8 \int_{1-2\varepsilon}^1 (1-r)^{\frac{q+d}{2}-s-1}\, dr = C_8(2\varepsilon)^{\frac{q+d}{2}-s},$$

where $C_8 > 0$. We conclude that, if $d-1 < s < \min(d, (q+d)/2)$, with $q > d-2$, there exist positive constants $\varepsilon_0 \le 1/4$, C_9, and β such that for any $0 < \varepsilon \le \varepsilon_0$ and any $|1-t| < \varepsilon$, we have

$$0 \le \int_{1-2\varepsilon}^1 v_s(t,r)(1-r^2)^{\frac{q-d}{2}}\, dr \le C_9 \left(|1-t|^{\frac{q+d}{2}-s} + \varepsilon^\beta\right).$$

Using this estimate it is easy to show that, under these assumptions, the potential is continuous also on ∂B_1.

The case $s = d-1$ follows analogously, by using (6.2) instead of (6.3) in the estimate of ${}_2F_1$ for $\alpha \sim 1$. ☐

Remark 7.3.5. From Lemma 7.3.4, we infer that $\mathcal{W}_s * \mu_q^E \in C^0(\mathbb{R}^d)$ in particular when $q = s \in [d-2, d) \cap (0, d)$, and when $q = s+2$, with $s \in [d-4, d) \cap (0, d)$. Such a continuity result will be relevant to extend up to the boundary of E the formulas (7.24) and (7.25) of the following Theorem 7.3.6.

We are now in a position to prove the Fourier representation of the potential $\mathcal{W}_s * \mu_q^E$ inside E.

Theorem 7.3.6. *Let $s \in (0, d)$ and $q \in [d-2, +\infty)$. Let $\mathcal{W}_s$ be as in (7.1) with $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ a continuous, even, and strictly positive function. Suppose that $\widehat{\mathcal{W}_s}$ is continuous on $\mathbb{R}^d \setminus \{0\}$. Let $E := RD\overline{B}_1$ be an ellipsoid, and let μ_q^E be the push-forward of the measure μ_q by the map T^E. Then,*

$$(\mathcal{W}_s * \mu_q^E)(x) = \tilde{c}_{d,s,q} \int_{\mathbb{S}^{d-1}} \frac{\Psi(\omega)}{|DR^T\omega|^s} \, {}_2F_1\left(\frac{s-q}{2}, \frac{s}{2}; \frac{1}{2}; \alpha^2(x,\omega)\right) d\mathcal{H}^{d-1}(\omega) \tag{7.22}$$

for every x in the interior of E, where $\Psi := \widehat{\mathcal{W}_s}|_{\mathbb{S}^{d-1}}$,

$$\tilde{c}_{d,s,q} := \frac{2^{s-d-1}\Gamma(1+\frac{q}{2})\Gamma(\frac{s}{2})}{\pi^d \Gamma(1+\frac{q-s}{2})}, \tag{7.23}$$

and

$$\alpha(x,\omega) := \frac{x \cdot \omega}{|DR^T\omega|}.$$

In particular, for $q = s$ the potential function $\mathcal{W}_s * \mu_s^E$ is constant in E and is given by

$$(\mathcal{W}_s * \mu_s^E)(x) = \frac{2^{s-d-2}}{\pi^d} s\Gamma^2\left(\frac{s}{2}\right) \int_{\mathbb{S}^{d-1}} \frac{\Psi(\omega)}{|DR^T\omega|^s} \, d\mathcal{H}^{d-1}(\omega) \tag{7.24}$$

for every $x \in E$. Finally, for $s \in [d-4, d) \cap (0, d)$ and $q = s+2$, the potential function $\mathcal{W}_s * \mu_{s+2}^E$ is, up to an additive constant, a quadratic function in E given by

$$\begin{aligned}(\mathcal{W}_s * \mu_{s+2}^E)(x) &= \tilde{c}_{d,s,s+2} \int_{\mathbb{S}^{d-1}} \frac{\Psi(\omega)}{|DR^T\omega|^s} \, d\mathcal{H}^{d-1}(\omega) \\ &\quad - s\,\tilde{c}_{d,s,s+2} \int_{\mathbb{S}^{d-1}} \frac{\Psi(\omega)}{|DR^T\omega|^{s+2}} (x\cdot\omega)^2 \, d\mathcal{H}^{d-1}(\omega)\end{aligned} \tag{7.25}$$

for every $x \in E$.

Remark 7.3.7. Theorem 7.3.6 provides an alternative way to derive [Fra+25, Equation (2.31)] and in fact extends it to the entire range of $s \in (\max\{d-4, 0\}, d)$ in any space dimension d.

Proof of Theorem 7.3.6. Formula (7.22) is a consequence of the Fourier Inversion Theorem. However, we cannot apply the inversion formula directly to $\mathcal{W}_s * \mu_q^E$, since its Fourier's transform fails to be integrable at infinity for $2s - q \geq 1$. Indeed, by Lemmas 6.2.6, 6.2.7 and 7.3.3, and due to the asymptotic behaviour (6.8) of the tail of the Bessel function at infinity, we have that

$$\widehat{\mathcal{W}_s * \mu_q^E} = O(|\xi|^{s-d-\frac{q}{2}-\frac{1}{2}}), \qquad |\xi| \to \infty.$$

We thus proceed by approximation. For $r > 0$ let B_r be the ball of radius r centred at the origin and let χ_{B_r} denote its characteristic function. We set $\chi_r := |B_r|^{-1}\chi_{B_r}$ and define $P_r := \chi_r * (\mathcal{W}_s * \mu_q^E)$.

By Lemma 7.3.4, P_r belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and P_r converges pointwise in $\mathbb{R}^d \setminus \partial E$ (thus almost everywhere in $\mathbb{R}^d$) to $\mathcal{W}_s * \mu_q^E$, as $r \to 0^+$. If

$$0 < s < \min\left(d, \frac{q+d}{2}\right),$$

the function P_r is continuous on $\mathbb{R}^d$, and P_r converges pointwise in $\mathbb{R}^d$ to $\mathcal{W}_s * \mu_q^E$, as $r \to 0^+$.

Note that, by the controlled growth behaviour at infinity, $\mathcal{W}_s$ is a tempered distribution. In particular, since μ_q has compact support, $\mathcal{W}_s * \mu_q^E$ is still a tempered distribution (see Section 6.2 for details). Lastly, since χ_r has compact support, P_r is a tempered distribution and $\widehat{P}_r = \widehat{\chi}_r\, \widehat{\mathcal{W}_s}\, \widehat{\mu}_q^E$. To compute $\widehat{\chi}_r$ we apply Formula 6.2.5 and obtain

$$\widehat{\chi}_r(\xi) = \frac{(2\pi)^{\frac{d}{2}}}{|B_r||\xi|^{\frac{d}{2}-1}} \int_0^r \rho^{\frac{d}{2}} J_{\frac{d}{2}-1}(\rho|\xi|)\, d\rho = \frac{2^{\frac{d}{2}}\Gamma(1+\frac{d}{2})}{r^{\frac{d}{2}}} \frac{1}{|\xi|^{\frac{d}{2}}} J_{\frac{d}{2}}(r|\xi|), \tag{7.26}$$

where we used that $|B_r| = r^d \pi^{\frac{d}{2}} / \Gamma(1+\frac{d}{2})$.

By the homogeneity of $\widehat{\mathcal{W}_s}$ given by Lemmas 6.2.6, 6.2.7 and 7.3.3, and the asymptotic behaviour of Bessel functions in (6.7)–(6.8), we deduce that

$$\begin{aligned} \widehat{P}_r(\xi) &= O(|\xi|^{s-d}), && |\xi| \to 0^+, \\ \widehat{P}_r(\xi) &= O(|\xi|^{s-\frac{q}{2}-\frac{3}{2}d-1}), && |\xi| \to +\infty. \end{aligned}$$

Since $2s - q < d + 2$ in our setting, this implies that $\widehat{P}_r \in L^1(\mathbb{R}^d)$. Thus, we can apply the Fourier Inversion Theorem 6.2.1 (see also Remark 6.2.2). By Lemma 7.3.3, (7.16), and (7.26) we obtain

$$P_r(x) = \frac{\tilde{c}_{q,d}}{r^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{1}{|\xi|^{\frac{3}{2}d-s}|DR^T\xi|^{\frac{q}{2}}} J_{\frac{d}{2}}(r|\xi|) J_{\frac{q}{2}}(|DR^T\xi|)\Psi\left(\frac{\xi}{|\xi|}\right) \cos(x\cdot\xi)\, d\xi, \tag{7.27}$$

where

$$\tilde{c}_{q,d} := \frac{2^{\frac{q-d}{2}}}{\pi^d}\Gamma\left(1+\frac{q}{2}\right)\Gamma\left(1+\frac{d}{2}\right).$$

In (7.27) the imaginary part can be dropped because $\widehat{\chi}_r\, \widehat{\mathcal{W}_s}\, \widehat{\mu}_q^E$ is even. Writing (7.27) in polar coordinates yields that $P_r(x)$ is equal to

$$\frac{\tilde{c}_{q,d}}{r^{\frac{d}{2}}} \int_{\mathbb{S}^{d-1}} \frac{\Psi(\omega)}{|DR^T\omega|^{\frac{q}{2}}} \int_0^\infty \rho^{s-\frac{q}{2}-\frac{d}{2}-1} J_{\frac{d}{2}}(r\rho) J_{\frac{q}{2}}(\rho|DR^T\omega|) \cos(\rho\omega\cdot x)\, d\rho\, d\mathcal{H}^{d-1}(\omega).$$

Set

$$t := |DR^T\omega|\rho, \quad \beta(r,\omega) := \frac{r}{|DR^T\omega|}, \quad \text{and} \quad \alpha(x,\omega) := \frac{x\cdot\omega}{|DR^T\omega|}. \tag{7.28}$$

Changing variables in the integral with respect to ρ we obtain

$$P_r(x) = \frac{\tilde{c}_{q,d}}{r^{\frac{d}{2}}}\int_{\mathbb{S}^{d-1}} \frac{\Psi(\omega)}{|DR^T\omega|^{s-\frac{d}{2}}} I_r(x,\omega)\, d\mathcal{H}^{d-1}(\omega), \tag{7.29}$$

where

$$I_r(x,\omega) := \int_0^\infty t^{s-\frac{q}{2}-\frac{d}{2}-1} J_{\frac{d}{2}}(t\beta(r,\omega)) J_{\frac{q}{2}}(t)\cos(t\alpha(x,\omega))\, dt.$$

To compute I_r we recall identity (6.9), that gives

$$\cos(t\alpha(x,\omega)) = \cos(t|\alpha(x,\omega)|) = \sqrt{\frac{\pi}{2}}\, t^{\frac{1}{2}}|\alpha(x,\omega)|^{\frac{1}{2}} J_{-\frac{1}{2}}(t|\alpha(x,\omega)|),$$

where the last expression on the right-hand side is extended at $t\alpha(x,\omega) = 0$ by continuity given the asymptotic (6.7). Thus, we can rewrite I_r as

$$\begin{aligned} I_r(x,\omega) = {} & \sqrt{\frac{\pi}{2}}|\alpha(x,\omega)|^{\frac{1}{2}} \\ & \times \int_0^\infty t^{s-\frac{q}{2}-\frac{d}{2}-\frac{1}{2}} J_{\frac{d}{2}}(t\beta(r,\omega)) J_{\frac{q}{2}}(t) J_{-\frac{1}{2}}(t|\alpha(x,\omega)|)\, dt. \end{aligned} \tag{7.30}$$

For the integral in (7.30) we use Formula 6.1.4. For a fixed x in the interior of E, $r > 0$ and $\omega \in \mathbb{S}^{d-1}$ we apply the Formula 6.1.4 with

$$\begin{aligned} &\lambda = s - \frac{q-d+1}{2}, \quad \nu = -\frac{1}{2}, \quad \mu = \frac{d}{2}, \\ &\rho = \frac{q}{2}, \quad a = |\alpha(x,\omega)|, \quad b = \beta(r,\omega), \quad c = 1. \end{aligned}$$

Hence, conditions (6.10) translates into

$$s > 0, \quad 2s - q < d + 4, \quad \text{and} \quad |\alpha(x,\omega)| < 1 - \beta(r,\omega).$$

The first two conditions are trivially satisfied for s and q in the range under consideration. As for the last one, we observe that $x = pRD\eta$ for some $\eta \in \mathbb{S}^{d-1}$ and $0 \le p < 1$, and thus

$$|\alpha(x,\omega)| = \frac{|x\cdot\omega|}{|DR^T\omega|} = \frac{p|\eta\cdot DR^T\omega|}{|DR^T\omega|} \le p < 1$$

for every $\omega \in \mathbb{S}^{d-1}$. Since $\beta(r,\omega) \to 0$ as $r \to 0^+$, uniformly with respect to $\omega \in \mathbb{S}^{d-1}$, there exist $\delta \in (0,1)$ and $r_0 > 0$ such that for $r < r_0$ we have

$$|\alpha(x,\omega)| + \beta(r,\omega) \le \delta \qquad \forall\, \omega \in \mathbb{S}^{d-1}.$$

For x in the interior of E, $r < r_0$ and $\omega \in \mathbb{S}^{d-1}$ we can then evaluate the integral in (7.30), and deduce that

$$\begin{aligned} I_r(x,\omega) &= \sqrt{\frac{\pi}{2}}|\alpha(x,\omega)|^{\frac{1}{2}} \frac{2^{s-\frac{q}{2}-\frac{d}{2}-\frac{1}{2}}\beta^{\frac{d}{2}}(r,\omega)|\alpha(x,\omega)|^{-\frac{1}{2}}\Gamma(\frac{s}{2})}{\Gamma(\frac{d}{2}+1)\Gamma(\frac{1}{2})\Gamma(1+\frac{q-s}{2})} \\ &\quad \times F_4\Big(\frac{s-q}{2}, \frac{s}{2}; \frac{d}{2}+1, \frac{1}{2}; \beta^2(r,\omega), \alpha^2(x,\omega)\Big) \\ &= \frac{2^{s-\frac{q}{2}-\frac{d}{2}-1}\beta^{\frac{d}{2}}(r,\omega)\Gamma(\frac{s}{2})}{\Gamma(\frac{d}{2}+1)\Gamma(1+\frac{q-s}{2})} F_4\Big(\frac{s-q}{2}, \frac{s}{2}; \frac{d}{2}+1, \frac{1}{2}; \beta^2(r,\omega), \alpha^2(x,\omega)\Big), \end{aligned}$$

where we have used property (iii) of the Gamma function in Section 6.1. By (7.29) and the definition of β in (7.28) we conclude that, for x in the interior of E and $r < r_0$, $P_r(x)$ is given by

$$\tilde{c}_{d,s,q} \int_{\mathbb{S}^{d-1}} \frac{\Psi(\omega)}{|DR^T\omega|^s} F_4\Big(\frac{s-q}{2}, \frac{s}{2}; \frac{d}{2}+1, \frac{1}{2}; \beta^2(r,\omega), \alpha^2(x,\omega)\Big)\, d\mathcal{H}^{d-1}(\omega), \tag{7.31}$$

where $\tilde{c}_{d,s,q}$ is the constant in (7.23).

We now pass to the limit as $r \to 0^+$. By (6.6) and by the definition of β we have

$$\lim_{r\to 0} F_4\Big(\frac{s-q}{2}, \frac{s}{2}; \frac{d}{2}+1, \frac{1}{2}; \beta^2(r,\omega), \alpha^2(x,\omega)\Big) = {}_2F_1\Big(\frac{s-q}{2}, \frac{s}{2}; \frac{1}{2}; \alpha^2(x,\omega)\Big). \tag{7.32}$$

Moreover, since F_4 is analytic in its domain of definition and

$$|\alpha(x,\omega)| + \beta(r,\omega) \le \delta < 1 \quad \text{for } r < r_0,$$

the convergence (7.32) is uniform with respect to $\omega \in \mathbb{S}^{d-1}$. Hence, passing to the limit in (7.31) and recalling that P_r converges to $\mathcal{W}_s * \mu_q^E$ pointwise in the interior of E, as $r \to 0^+$, we obtain (7.22) for every x in the interior of E.

The statement (7.24) for $q = s$ follows immediately in the interior of E from (7.22), owing to (6.4) and property (i) of the Gamma function in Section 6.1. Similarly, the statement (7.25) for $q = s + 2$ follows immediately in the interior of E from (7.22), owing to (6.5). In fact both formulas hold up to the boundary of E since in both cases the potential $\mathcal{W}_s * \mu_q^E$ is continuous, as observed in Remark 7.3.5. □

We conclude this section by proving Theorem 7.1.1.

Proof of Theorem 7.1.1. By Theorem 7.2.1 the minimizer of I_s^E exists, is unique, and is the unique measure satisfying the Euler–Lagrange conditions (EL1)–(EL3). Therefore, to conclude it is enough to show that μ_s^E satisfies (EL1)–(EL3). Condition (EL1) is trivially satisfied. Conditions (EL2)–(EL3) follow from Theorem 7.3.6 with $q = s$. □

7.4 The sub-Coulombic regime

We recall that in Section Section 7.3 we have shown that, for $s \ge d-2$, the minimizer μ_s^E of I_s^E is insensitive to the anisotropy Φ, and in particular if $E := \overline{B}_1$ we have $\mu_s^{\overline{B}_1} = \mu_{\mathrm{iso},s}$.

In this section we show that the equality above may fail for $s < d-2$ by providing an explicit example of a kernel $\mathcal{W}_s$ of the form (7.1) with continuous and non-negative Fourier's transform, for which $\mu_{\mathrm{iso},d-2}$ is not the minimizer of $I_s^{\overline{B}_1}$.

Lemma 7.4.1. *Let $d \geq 3$ and $s \in (0, d-2)$. Let $\mathcal{W}_s$ be as in (7.1) with $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ given by*

$$\Phi(\omega) := \sum_{i=1}^{d} \alpha_i \omega_i^2 \quad \textit{for } \omega \in \mathbb{S}^{d-1},$$

with $\alpha_i > 0$ for $i = 1, \dots, d$. Then $\widehat{\mathcal{W}_s}$ is non-negative on $\mathbb{S}^{d-1}$ if and only if

$$\alpha_i \leq \frac{1}{d-s-1} \sum_{j \neq i} \alpha_j \qquad \textit{for every } i = 1, \dots, d. \tag{7.33}$$

Proof. Let p_0 and $p_{2,i}$, for $i = 1, \dots, d$, be the homogeneous harmonic polynomials defined as

$$p_0 \equiv 1, \quad p_{2,i}(x) := (d-1)x_i^2 - \sum_{j \neq i} x_j^2,$$

for $x \in \mathbb{R}^d$. Then for $\omega \in \mathbb{S}^{d-1}$ we write Φ as

$$\Phi(\omega) = \frac{1}{d} \sum_{i=1}^{d} \alpha_i p_0(\omega) + \frac{1}{d} \sum_{i=1}^{d} \alpha_i p_{2,i}(\omega),$$

and hence, for $x \in \mathbb{R}^d$, $x \neq 0$, the kernel $\mathcal{W}_s$ can be rewritten as

$$\mathcal{W}_s(x) = \frac{1}{d} \sum_{i=1}^{d} \alpha_i \frac{p_0(x)}{|x|^s} + \frac{1}{d} \sum_{i=1}^{d} \alpha_i \frac{p_{2,i}(x)}{|x|^{s+2}}.$$

From Lemma 6.2.8 we deduce that for $\xi \in \mathbb{R}^d$, $\xi \neq 0$,

$$\begin{aligned} \widehat{\mathcal{W}_s}(\xi) &= \frac{1}{d} 2^{d-s} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{s}{2})} \frac{1}{|\xi|^{d-s}} \sum_{i=1}^{d} \alpha_i - \frac{1}{d} \sum_{i=1}^{d} \alpha_i 2^{d-s} \pi^{\frac{d}{2}} \frac{\Gamma(1+\frac{d-s}{2})}{\Gamma(1+\frac{s}{2})} \frac{p_{2,i}(\xi)}{|\xi|^{2+d-s}} \\ &= \frac{1}{d} 2^{d-s} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{s}{2})} \frac{1}{|\xi|^{d-s}} \sum_{i=1}^{d} \left(\alpha_i - \alpha_i \frac{d-s}{s} p_{2,i}\left(\frac{\xi}{|\xi|} \right) \right), \end{aligned} \tag{7.34}$$

where we have used property (i) of the Gamma function in Section 6.1. In particular, for $\omega \in \mathbb{S}^{d-1}$ we have

$$\begin{aligned} \frac{1}{d} \sum_{i=1}^{d} \left(\alpha_i - \alpha_i \frac{d-s}{s} p_{2,i}(\omega) \right) &= \frac{1}{d} \sum_{i=1}^{d} \left(\alpha_i - \alpha_i \frac{d-s}{s} (d\omega_i^2 - 1) \right) \\ &= \frac{1}{s} \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{d} \omega_j^2 - \frac{1}{s} \sum_{i=1}^{d} \alpha_i (d-s) \omega_i^2 \\ &= \frac{1}{s} \sum_{i=1}^{d} \left((1-d+s)\alpha_i + \sum_{j \neq i} \alpha_j \right) \omega_i^2. \end{aligned}$$

Thus, (7.34) can be rewritten as

$$\widehat{\mathcal{W}}_s(\xi) = \frac{1}{|\xi|^{d-s}} 2^{d-s} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{s}{2})} \frac{1}{s} \sum_{i=1}^{d} \left((1-d+s)\alpha_i + \sum_{j \neq i} \alpha_j \right) \frac{\xi_i^2}{|\xi|^2}.$$

Since $1-d+s<0$ for $s<d-2$, we conclude that $\widehat{\mathcal{W}}_s \geq 0$ if and only if (7.33) is satisfied. □

We are now in a position to prove the main result of this section.

Theorem 7.4.2. *Let $d \geq 3$ and $s \in (0, d-2)$. Let $\mathcal{W}_s$ be as in (7.1) with $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$ given by*

$$\Phi(x) := \frac{d-1}{d-s-1} x_1^2 + \sum_{i=2}^{d} x_i^2 = 1 + \frac{s}{d-s-1} x_1^2 \quad \text{for } x \in \mathbb{S}^{d-1}.$$

Then Φ is continuous, even, and strictly positive, and $\widehat{\mathcal{W}}_s$ is continuous and non-negative on $\mathbb{S}^{d-1}$. However, the measure $\mu_{\mathrm{iso},d-2}$ is not the minimizer of the energy $I_s^{\overline{B}_1}$ with kernel $\mathcal{W}_s$.

Proof. All the properties of Φ and $\mathcal{W}_s$ are straightforward, except for the sign of $\widehat{\mathcal{W}}_s$. By Lemma 7.4.1 the non-negativity of $\widehat{\mathcal{W}}_s$ on $\mathbb{S}^{d-1}$ is equivalent to

$$1 \leq \frac{1}{d-s-1} \left(d-2 + \frac{d-1}{d-s-1} \right). \tag{7.35}$$

A simple computation shows that (7.35) reduces to $s^2 - sd \leq 0$, which is true for $0 < s < d-2$.

To prove that the measure $\mu_{\mathrm{iso},d-2}$ is not the minimizer of $I_s^{\overline{B}_1}$, it is enough to show that

$$A := (\mathcal{W}_s * \mu_{\mathrm{iso},d-2})(e_2) - (\mathcal{W}_s * \mu_{\mathrm{iso},d-2})(e_1) \neq 0. \tag{7.36}$$

Indeed, since $\mathcal{W}_s * \mu_{\mathrm{iso},d-2} \in C^0(\mathbb{R}^d)$ by Lemma 7.3.4, equation (7.36) implies that $\mathcal{W}_s * \mu_{\mathrm{iso},d-2}$ is not constant $\mathcal{H}^{d-1}$-a.e. on ∂B_1, contradicting (EL2).

We note that

$$(\mathcal{W}_s * \mu_{\mathrm{iso},d-2})(e_1) = c_{d-2,d} \int_{\mathbb{S}^{d-1}} \frac{1}{|e_1 - \omega|^s} \left(1 + \frac{s}{d-s-1} \frac{(1-\omega_1)^2}{|e_1-\omega|^2} \right) d\mathcal{H}^{d-1}(\omega)$$

and by the change of variables $\tilde{\omega} := (\omega_2, \omega_1, \omega_3, \dots, \omega_d)$ we can rewrite

$$\begin{aligned}(\mathcal{W}_s * \mu_{\mathrm{iso},d-2})(e_2) &= c_{d-2,d} \int_{\mathbb{S}^{d-1}} \frac{1}{|e_2 - \omega|^s} \left(1 + \frac{s}{d-s-1} \frac{\omega_1^2}{|e_2-\omega|^2} \right) d\mathcal{H}^{d-1}(\omega) \\ &= c_{d-2,d} \int_{\mathbb{S}^{d-1}} \frac{1}{|e_1 - \tilde{\omega}|^s} \left(1 + \frac{s}{d-s-1} \frac{\tilde{\omega}_2^2}{|e_1-\tilde{\omega}|^2} \right) d\mathcal{H}^{d-1}(\tilde{\omega}).\end{aligned}$$

Therefore,

$$A = c_{d-2,d} \frac{s}{d-s-1} \int_{\mathbb{S}^{d-1}} \frac{\omega_2^2 - (1-\omega_1)^2}{|e_1-\omega|^{s+2}} d\mathcal{H}^{d-1}(\omega).$$

Since $|e_1-\omega|^2 = 2-2\omega_1$ for $\omega \in \mathbb{S}^{d-1}$, we have

$$|e_1-\omega|^{s+2} = 2^{\frac{s+2}{2}}(1-\omega_1)^{\frac{s+2}{2}} \qquad \text{for } \omega \in \mathbb{S}^{d-1}.$$

Hence, the claim (7.36) reduces to showing that

$$\int_{\mathbb{S}^{d-1}} \frac{\omega_2^2-(1-\omega_1)^2}{(1-\omega_1)^{\frac{s+2}{2}}}\, d\mathcal{H}^{d-1}(\omega) \neq 0. \tag{7.37}$$

We consider separately the two cases $d \geq 4$ and $d=3$. We start with the case $d \geq 4$. Passing to spherical coordinates, the integral in (7.37) can be rewritten as $C(d)(I_1-I_2)$, where $C(d)>0$ is a dimensional constant, and

$$\begin{aligned} I_1 &:= \int_0^\pi\int_0^\pi \frac{\sin^d(\varphi_1)\cos^2(\varphi_2)\sin^{d-3}(\varphi_2)}{(1-\cos(\varphi_1))^{\frac{s+2}{2}}}\, d\varphi_1 d\varphi_2,\\ I_2 &:= \int_0^\pi\int_0^\pi (1-\cos(\varphi_1))^{\frac{2-s}{2}}\sin^{d-2}(\varphi_1)\sin^{d-3}(\varphi_2)\, d\varphi_1 d\varphi_2. \end{aligned}$$

To conclude we need to show that $I_1-I_2\neq 0$. We write $I_1 = I_3 I_4$, where

$$\begin{aligned} I_3 &:= \int_0^\pi \sin^d(\varphi_1)(1-\cos(\varphi_1))^{\frac{-2-s}{2}}\, d\varphi_1,\\ I_4 &:= \int_0^\pi \cos^2(\varphi_2)\sin^{d-3}(\varphi_2)\, d\varphi_2. \end{aligned}$$

By the sine and the cosine duplication formula and by Formula 6.1.1 we obtain

$$\begin{aligned} I_3 &= 2^{d-\frac{s+2}{2}}\int_0^\pi \sin^{d-2-s}\left(\frac{\varphi_1}{2}\right)\cos^d\left(\frac{\varphi_1}{2}\right) d\varphi_1\\ &= 2^{d+1-\frac{s+2}{2}}\int_0^{\frac{\pi}{2}} \sin^{d-2-s}(\varphi_1)\cos^d(\varphi_1)\, d\varphi_1\\ &= 2^{d-1-\frac{s}{2}}B\Big(\frac{d-s-1}{2},\frac{d+1}{2}\Big), \end{aligned} \tag{7.38}$$

whereas

$$I_4 = 2\int_0^{\frac{\pi}{2}} \cos^2(\varphi_2)\sin^{d-3}(\varphi_2)\, d\varphi_2 = B\Big(\frac{d-2}{2},\frac{3}{2}\Big). \tag{7.39}$$

Similarly, we write $I_2 = I_5 I_6$, where

$$\begin{aligned} I_5 &:= \int_0^\pi (1-\cos(\varphi_1))^{\frac{2-s}{2}}\sin^{d-2}(\varphi_1)\, d\varphi_1\\ &= 2^{d-\frac{s}{2}}\int_0^{\frac{\pi}{2}} \sin^{d-s}(\varphi_1)\cos^{d-2}(\varphi_1)\, d\varphi_1 = 2^{d-1-\frac{s}{2}}B\Big(\frac{d-s+1}{2},\frac{d-1}{2}\Big) \end{aligned} \tag{7.40}$$

and

$$\begin{aligned} I_6 &:= \int_0^\pi \sin^{d-3}(\varphi_2)\, d\varphi_2 = 2^{d-2}\int_0^{\frac{\pi}{2}} \sin^{d-3}(\varphi_2)\cos^{d-3}(\varphi_2)\, d\varphi_2\\ &= 2^{d-3}B\Big(\frac{d-2}{2},\frac{d-2}{2}\Big). \end{aligned} \tag{7.41}$$

Combining (7.38)–(7.41) yields

$$I_1 - I_2 = I_3 I_4 - I_5 I_6 = 2^{d-1-\frac{s}{2}} B\left(\frac{d-s-1}{2}, \frac{d+1}{2}\right) B\left(\frac{d-2}{2}, \frac{3}{2}\right) \\ - 2^{2d-4-\frac{s}{2}} B\left(\frac{d-s+1}{2}, \frac{d-1}{2}\right) B\left(\frac{d-2}{2}, \frac{d-2}{2}\right).$$

By the definition of Beta function, the previous equality can be rewritten as

$$2^{d-1-\frac{s}{2}} \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(d-\frac{s}{2})} \left(\Gamma\Big(\frac{d-s-1}{2}\Big)\Gamma\Big(\frac{3}{2}\Big) - 2^{d-3} \frac{\Gamma(\frac{d-s+1}{2})\Gamma(\frac{d-1}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(d-2)} \right).$$

By properties (i) and (iii) of the Gamma function we have that

$$\Gamma\Big(\frac{d-s+1}{2}\Big) = \Big(\frac{d-s-1}{2}\Big)\Gamma\Big(\frac{d-s-1}{2}\Big)$$

and

$$\Gamma\Big(\frac{3}{2}\Big) = \frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) = \frac{\sqrt{\pi}}{2.}$$

Therefore, $I_1 - I_2 = 0$ if and only if

$$\frac{\sqrt{\pi}}{2}\Gamma(d-2) - 2^{d-4}(d-s-1)\,\Gamma\Big(\frac{d-1}{2}\Big)\Gamma\Big(\frac{d}{2}-1\Big) = 0. \tag{7.42}$$

Finally, properties (ii) of the Gamma function gives

$$\Gamma\Big(\frac{d-1}{2}\Big)\Gamma\Big(\frac{d}{2}-1\Big) = 2^{3-d}\sqrt{\pi}\Gamma(d-2),$$

hence (7.42) reduces to

$$\frac{\sqrt{\pi}}{2}\Gamma(d-2)(2-d+s) = 0,$$

which is never satisfied for $s < d-2$. This concludes the proof for $d \geq 4$. When $d = 3$, the previous computations can be repeated with the only difference being that the integration interval for φ_2 is $(0, 2\pi)$, instead of $(0, \pi)$. This change simply introduces an extra factor of 2 in the expressions for both I_4 and I_6, so the same calculations still lead to the desired conclusion. □

A simple continuity argument leads to the following.

Corollary 7.4.3. *Let $d \geq 3$ and $s \in (0, d-2)$. For any $\varepsilon > 0$, we set*

$$\Phi_\varepsilon(\omega) := \frac{d-1-\varepsilon}{d-s-1}\omega_1^2 + \sum_{i=2}^{d} \omega_i^2 \quad \text{for } \omega \in \mathbb{S}^{d-1}$$

and for $x \in \mathbb{R}^d$, $x \neq 0$,

$$\mathcal{W}_{s,\varepsilon}(x) := \frac{1}{|x|^s}\Phi_\varepsilon\Big(\frac{x}{|x|}\Big).$$

Then there exists $\varepsilon_0 > 0$, depending on s, such that for any $0 < \varepsilon \leq \varepsilon_0$ the profile Φ_ε is continuous, even, and strictly positive, $\widehat{\mathcal{W}_{s,\varepsilon}}$ is continuous and strictly positive on $\mathbb{S}^{d-1}$, but the measure $\mu_{\mathrm{iso},d-2}$ is not the minimizer of the energy $I_s^{\overline{B}_1}$ with kernel $\mathcal{W}_s$ replaced by $\mathcal{W}_{s,\varepsilon}$.

8 Optimal constructions of grain boundaries

8.1 Assumptions and main results

In this chapter we consider the two-dimensional semi-discrete dislocation energy introduced by Lauteri and Luckhaus in [LL16] (and later studied in [FGS25]) and propose an alternative, simpler, and more natural construction for the grain boundary between two crystal grains with small orientation difference, which works for a general Bravais' lattice.

Let $L > 0$, and let $\Omega := [-L, L] \times [-2L, 0]$ represent the two-dimensional cross-section of a three-dimensional crystal. Let $\varepsilon > 0$ be a small parameter representing the microscopic scale (e.g. the size of the lattice cell), and let $\tau, \lambda > 0$ be two parameters representing the (rescaled) length of the Burgers' vector and the (rescaled) size of the core-region around a dislocation, respectively. Consider $\tilde{\mathcal{B}}$ a Bravais' lattice in $\mathbb{R}^2$, and $\tilde{b}_1, \tilde{b}_2 \in \mathbb{R}^2$ a basis of the lattice, so that $\tilde{\mathcal{B}} = \operatorname{span}_{\mathbb{Z}^2}\{\tilde{b}_1, \tilde{b}_2\}$, and $\operatorname{span}_{\mathbb{R}^2}\{\tilde{b}_1, \tilde{b}_2\} = \mathbb{R}^2$. Let $\phi \in [0, 2\pi]$ denote the lattice orientation, so that the rotated lattice is $\mathcal{B} := R_\phi \tilde{\mathcal{B}}$, where $R_\phi \in \mathrm{SO}(2)$ is the rotation

$$R_\phi := \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Clearly $\mathcal{B}$ is the lattice generated by $b_1 := R_\phi \tilde{b}_1$ and $b_2 := R_\phi \tilde{b}_2$, which we write for simplicity as

$$b_1 := \begin{pmatrix} \cos\eta \\ \sin\eta \end{pmatrix}, \quad b_2 := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

We assume that $\sin\eta \neq 0$, that is $b_1 \neq e_1$. For $b_1 = e_1$ one can construct the grain boundary in a simpler way (see Remark 8.3.2). Finally, we denote with $\alpha > 0$ the small misorientation between two neighbouring grains occupying the regions $[-L, 0] \times [-2L, 0]$ and $[0, L] \times [-2L, 0]$.

We define the class of admissible strains $\mathcal{C}_\varepsilon(\Omega)$ as follows:

$$\mathcal{C}_\varepsilon(\Omega) := \Big\{(\beta, S) : \beta \in L^1(\Omega;\mathbb{R}^{2\times 2}) \cap L^2(\Omega \setminus \overline{B_{\lambda\varepsilon}(S)};\mathbb{R}^{2\times 2}),$$
$$S \text{ relatively closed, satisfying (H1)–(H3)}\Big\},$$

where

(H1) $\operatorname{supp}(\operatorname{curl}\beta) \subset S$,

(H2) $\langle T_t^\gamma(\beta), 1\rangle \in \tau\varepsilon\mathcal{B}$ for every simple, closed, and Lipschitz curve γ in $\Omega \setminus \overline{B_{\lambda\varepsilon}(S)}$, where T_t^γ is the tangent trace on γ,

(H3) β satisfies the symmetric boundary conditions

$$\beta = \begin{cases} R_{-\alpha} & \text{in } [-L, -L+\ell] \times [-2L, 0], \\ R_\alpha & \text{in } [L-\ell, L] \times [-2L, 0], \end{cases}$$

with $0 < \ell \ll L$.

Here,

$$B_{\lambda\varepsilon}(S) = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, S) < \lambda\varepsilon\}.$$

Condition (H2) deserves some clarification. In particular, we should illustrate why T_t^γ is well-defined. Since $\gamma \subset \Omega \setminus \overline{B_\varepsilon(S)}$, we have that $\operatorname{dist}(\gamma, B_\varepsilon(S)) > 0$. Then, one can show that there exists an open set $A \subset \Omega \setminus \overline{B_\varepsilon(S)}$ such that $\partial A = \gamma \cup \tilde{\gamma}$, with $\tilde{\gamma}$ being a smooth curve disjoint from γ. By (H1), $\operatorname{curl}(\beta) = 0$ in A, thus

$$\beta \in L^2_{\text{curl}}(A;\mathbb{R}^{2\times 2}).$$

It follows that β admits a tangent trace in A (see Section 6.5). Since ∂A is made of two disjoint connected component, one of them being γ, we can identify the tangent trace of β on A with the sum of two elements, one of them being the tangent trace on γ (see Section 6.5). One can also show that the trace on γ obtained as above is independent of the choice of the set A, thus, $T_t^\gamma(\beta)$ is well-defined.

Condition (H2) is a quantization property of the averaged macroscopic Burgers' vector. Indeed, if in addition $\beta \in C^0(\Omega;\mathbb{R}^2)$, (H2) can be rewritten as

$$\int_\gamma \beta t \, d\mathcal{H}^1 \in \tau\varepsilon\mathcal{B},$$

and, if $\operatorname{curl}(\beta) \in L^2(\Omega;\mathbb{R}^2)$, then by Stokes' Theorem (see Section 6.5)

$$\int_\Gamma \operatorname{curl}(\beta) \, dx = \int_\gamma \beta t \, d\mathcal{H}^1 \in \tau\varepsilon\mathcal{B},$$

where Γ is the set enclosed by γ and the left-hand side is the averaged macroscopic Burgers' vector.

The energy associated to an admissible pair $(\beta, S) \in \mathcal{C}_\varepsilon(\Omega)$ is

$$E_\varepsilon(\beta, S) := \frac{1}{\varepsilon}\left(\int_{\Omega\setminus B_{\lambda\varepsilon}(S)} \mathcal{W}(\beta) \, dx + \mathcal{L}^2(B_{\lambda\varepsilon}(S))\right), \tag{8.1}$$

with the first term being the elastic energy and the second term a core energy. We assume the elastic energy to have at most quadratic growth, namely there is a constant $C > 0$ such that

$$\mathcal{W}(M) \leq C\,\mathrm{dist}^2(M, SO(2)) \qquad \forall\, M \text{ in a neighbourhood of } \mathrm{SO}(2). \tag{8.2}$$

The energy is written in terms of the strain field β. However, outside S, β is locally the gradient of a deformation.

One of the goals in [LL16] was to show that, for ε small enough,

$$\inf\{E_\varepsilon(\beta, S) : (\beta, S) \in \mathcal{C}_\varepsilon(\Omega)\} \leq C\alpha|\log\alpha|,$$

with $C > 0$, as claimed by Read and Shockley in [RS50].

This has been done in [LL16] for $\mathcal{B} = \tilde{\mathcal{B}} = \mathbb{Z}^2$ (that is $\phi = 0$), and later, in [FGS25], generalized to the rotated lattice $R_\phi\mathbb{Z}^2$. Note that the case $\phi \neq 0$ is equivalent to having an unrotated lattice $\mathbb{Z}^2$, and *asymmetric* boundary conditions $R_{\pm\alpha-\phi}$, as considered in [RS50].

In this chapter we deal with the case of asymmetric grain boundaries for a general Bravais' lattice $\mathcal{B}$, and propose an alternative, simpler construction for the grain boundary. In our construction the deformation is piecewise affine on a finite polyhedral Cacciopoli's partition, hence the strain β is piecewise constant, and the boundaries of the regions of the partition are straight segments. Thus, our construction is admissible for the more restrictive minimization problem

$$\inf\{E_\varepsilon(\beta, S) : (\beta, S) \in \mathcal{C}_\varepsilon(\Omega),\ \beta \text{ piecewise constant}\}, \tag{8.3}$$

which can be recast purely in terms of matrices. Within this more restrictive framework, a competitor for (8.3) is a piecewise constant map β which connects the rotations $R_{\pm\alpha}$ on the two outer vertical strips of the domain, in such a way that the constant values of β are close to $SO(2)$ and that the constant matrices in the construction are almost always rank-one connected along their common, straight interface. Recall that two matrices M_1 and M_2 are said to be rank-one connected along a direction ν if $M_1 - M_2 = w \otimes \nu$ for some vector w.

We now briefly describe our construction. As in [LL16] and [FGS25], the field β agrees with the respective boundary conditions $R_{\pm\alpha}$ in the majority of the domain, except for a thin vertical strip of width of order ε/α. In the general case $\theta \neq 0$ this strip is composed of two vertical sub-strips next to each other, and only one Burgers vector is active per sub-strip. In each sub-strip, dislocations are arranged periodically, at a distance of order α/ε. The presence of a dislocation is incorporated by means of a jump of the deformation, that simulates the opening of the lattice to make space for the presence of the extra half plane of atoms carried by the defect. We take into account the accumulation of the Burgers' vector by making such opening wider and wider as we travel down in the lattice.

8.2 The grain boundary construction

In this section we present our construction, illustrated in Figure 8.1.

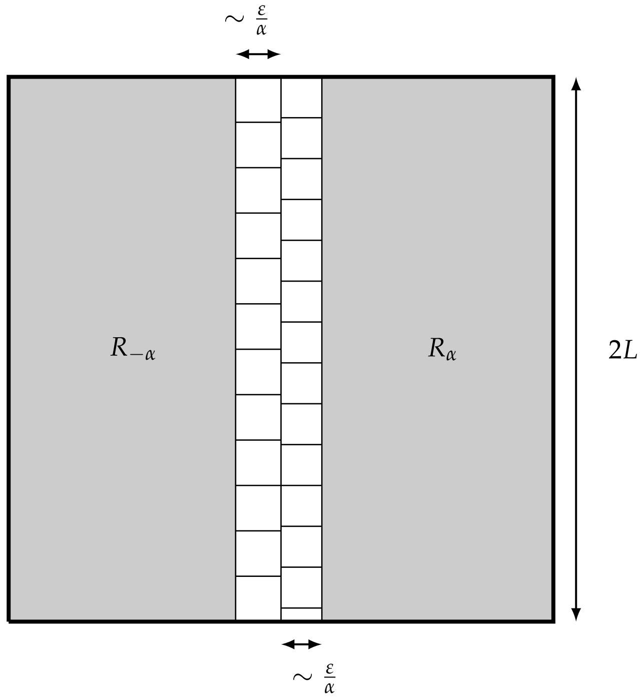

Figure 8.1: The vertical grain boundary.

8.2.1 The splitting of the domain

Let $\ell_1, \ell_2 > 0$ be positive parameters to be defined later on, representing the width of the two vertical sub-strips Σ_1 and Σ_2 defined as

$$\Sigma_1 := [-2\ell_1, 0] \times [-2L, 0], \quad \Sigma_2 := [0, 2\ell_2] \times [-2L, 0],$$

and let

$$\Sigma_{-\alpha} := [-L, -2\ell_1] \times [-2L, 0], \quad \Sigma_\alpha := [-2\ell_2, L] \times [-2L, 0]$$

be the strips where the boundary conditions $R_{\pm\alpha}$ will be imposed. Then we split the domain as

$$\Omega = \Sigma_{-\alpha} \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_\alpha.$$

The vertical sub-strips Σ_i are further split into squares with side-length $2\ell_i$. More precisely, we denote the generic square in the strip Σ_i as

$$Q_i^k := Q_i + t_i^k, \quad Q_i := [-\ell_i, \ell_i]^2, \quad k = 0, 1, 2, \dots, N_i,$$

where

$$t_i^k := -(2k+1)\ell_i e_2 + (-1)^i \ell_i e_1 \tag{8.4}$$

is a translation vector, and $N_i := \lceil L/\ell_i \rceil - 1$. Finally, we denote with $D_i^0 := [-r_{0,i}, r_{0,i}]^2$, and with D_i^n the family of dyadic annuli

$$D_i^n := [-r_{n,i}, r_{n,i}]^2 \setminus [-r_{n-1,i}, r_{n-1,i}]^2, \quad r_{n,i} := 2^n r_{0,i},$$

for $n = 1, \dots, \bar{n}_i$, with $\bar{n}_i$, $i = 1, 2$, such that

$$r_{\bar{n}_i,i} = \ell_i.$$

While l_i, and $\bar{n}_i$ are parameters to be fixed, $r_{0,i}$ is determined once they are chosen. By construction, we have

$$Q_i = [-r_{\bar{n}_i,i}, r_{\bar{n}_i,i}]^2 = \bigcup_{n=0}^{\bar{n}_i} D_i^n, \quad i = 1, 2.$$

For later convenience, we define the following subsets of D_i^n:

$$\begin{aligned}
\Delta_1^{a,n} &:= \mathrm{conv}(\{(r_{n-1,1}, -r_{n-1,1}), (r_{n,1}, r_{n,1}), (r_{n-1,1}, r_{n-1,1})\}),\\
\Delta_1^{b,n} &:= \mathrm{conv}(\{(r_{n,1}, r_{n,1}), (r_{n,1}, -r_{n,1}), (r_{n-1,1}, -r_{n-1,1})\}),\\
\Delta_2^{a,n} &:= \mathrm{conv}(\{(-r_{n-1,2}, -r_{n-1,2}), (-r_{n,2}, r_{n,2}), (-r_{n-1,2}, r_{n-1,2})\}),\\
\Delta_2^{b,n} &:= \mathrm{conv}(\{(-r_{n,2}, r_{n,2}), (-r_{n,2}, -r_{n,2}), (-r_{n-1,2}, -r_{n-1,2})\}),
\end{aligned}$$

where conv denotes the convex hull. Finally, we partition Q_1 and Q_2 in a slightly different way, namely

$$\begin{aligned}
Q_1 &= D_1^0 \cup Q_1^l \cup T_1^a \cup T_1^b \cup (\cup_{n=1}^{\bar{n}_1} \Delta_1^{a,n}) \cup (\cup_{n=1}^{\bar{n}_1} \Delta_1^{b,n}),\\
Q_2 &= D_2^0 \cup Q_2^r \cup T_2^a \cup T_2^b \cup (\cup_{n=1}^{\bar{n}_2} \Delta_2^{a,n}) \cup (\cup_{n=2}^{\bar{n}_1} \Delta_2^{b,n}),
\end{aligned}$$

where

$$\begin{aligned}
Q_1^l &:= [-r_{\bar{n}_1,1}, 0] \times [-r_{\bar{n}_1,1}, r_{\bar{n}_1,1}] \setminus D_1^0,\\
Q_2^r &:= [0, r_{\bar{n}_2,2}] \times [-r_{\bar{n}_2,2}, r_{\bar{n}_2,2}] \setminus D_2^0,
\end{aligned}$$

and

$$\begin{aligned}
T_1^a &:= \mathrm{conv}(\{(0, r_{0,2}), (r_{0,2}, r_{0,2}), (r_{\bar{n}_2,2}, r_{\bar{n}_2,2}), (0, r_{\bar{n}_2,2})\}),\\
T_1^b &:= \mathrm{conv}(\{(0, -r_{0,2}), (r_{0,2}, -r_{0,2}), (r_{\bar{n}_2,2}, -r_{\bar{n}_2,2}), (0, -r_{\bar{n}_2,2})\}),\\
T_2^a &:= \mathrm{conv}(\{(0, r_{0,1}), (-r_{0,1}, r_{0,1}), (-r_{\bar{n}_1,1}, r_{\bar{n}_1,1}), (0, r_{\bar{n}_1,1})\}),\\
T_2^b &:= \mathrm{conv}(\{(0, -r_{0,1}), (-r_{0,1}, -r_{0,1}), (-r_{\bar{n}_1,1}, -r_{\bar{n}_1,1}), (0, -r_{\bar{n}_1,1})\}).
\end{aligned}$$

These are the regions where the strain will be constant, forming a finite Caccioppoli partition of Q_i into polygonal domains (see Figure 8.2). All the translated squares Q_i^k are partitioned in the same way. We indicate with a further subscript k the translations of these sets by the vector t_i^k in (8.4), which are subsets of Q_i^k.

8.2.2 The construction of the piecewise constant strain

For $p_i \in \mathbb{R}^2$ and $v_i \in \mathbb{R}^2$ for $i = 1, 2, 3$, we define the affine interpolation I_Δ with values v_i at the points p_i as follows: for every $x \in \Delta := \mathrm{conv}(\{p_1, p_2, p_3\})$,

$$I_\Delta(x) := \sum_{i=1}^{3} v_i \Phi_i(x), \tag{8.5}$$

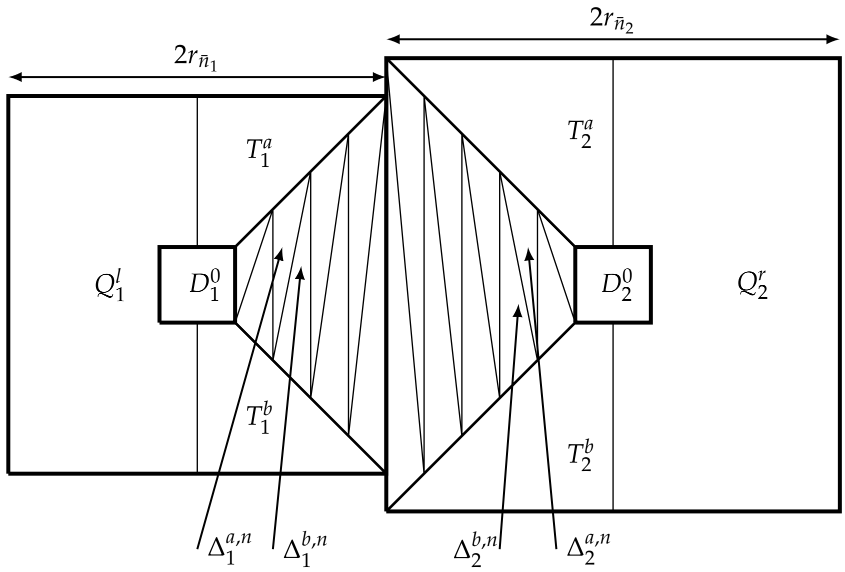

Figure 8.2: The regions of the construction.

where

$$
\begin{aligned}
&P := \begin{pmatrix} p_2 - p_1 & p_3 - p_1 \end{pmatrix}, \quad \Phi_i(x) := \Phi_i^{\mathrm{ref}}(P^{-1}(x - p_1)),\\
&\Phi_1^{\mathrm{ref}}(x) := 1 - x_1 - x_2, \quad \Phi_2^{\mathrm{ref}}(x) := x_1, \quad \Phi_3^{\mathrm{ref}}(x) := x_2.
\end{aligned}
$$

The constant gradient of the affine interpolation is given by

$$
\begin{aligned}
\nabla I_\Delta(x) &= \sum_{i=1}^{3} v_i \otimes \nabla \Phi_i(x)\\
&= v_1 \left(P^{-T} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right)^T + v_2 \left(P^{-T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)^T + v_3 \left(P^{-T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^T \qquad (8.6)\\
&= \frac{1}{\det P} \left(v_1 \otimes \begin{pmatrix} -P_{22} + P_{21} \\ P_{12} - P_{11} \end{pmatrix} + v_2 \otimes \begin{pmatrix} P_{22} \\ -P_{12} \end{pmatrix} + v_3 \otimes \begin{pmatrix} -P_{21} \\ P_{11} \end{pmatrix} \right),
\end{aligned}
$$

where P^{-T} denotes the matrix $(P^{-1})^T$. Clearly, if $v_i = w_i + t$ for $i = 1, 2, 3$, with a common translation vector $t \in \mathbb{R}^2$, then one can use w_i instead of v_i in (8.6).

We now define affine functions in all the components of Q_i^k as follows. Let $R_i := R_{\alpha_i}$, with $\alpha_i := (-1)^i \alpha$. In the top half of each square Q_i^k we impose a translation along b_i of k units, in order to accommodate all the dislocations with Burgers' vector b_i up until that point. In the bottom half instead we increment the translation of the top by one unit, to describe an additional dislocation. The regions will also undergo a rotation R_i to match the boundary conditions. More precisely we have the following.

Regions $Q^l_{1,k}$ and $Q^r_{2,k}$. We define deformations in these regions as the boundary conditions on the neighbouring $\Sigma_{\pm\alpha}$, that is

$$I_{Q^l_{1,k}}(x) := R_{-\alpha}x, \quad x \in Q^l_{1,k},$$
$$I_{Q^r_{2,k}}(x) := R_{\alpha}x, \quad x \in Q^r_{2,k}.$$

Clearly $\nabla I_{Q^l_{1,k}} = R_{-\alpha}$ and $\nabla I_{Q^r_{2,k}} = R_\alpha$, so that the strain is either R_α or $R_{-\alpha}$.

Regions $T^a_{i,k}$ and $T^b_{i,k}$. We define deformations in these regions as the affine maps

$$I_{T^a_{i,k}}(x) := R_i x - (-1)^i k b_i, \quad x \in T^a_{i,k},$$
$$I_{T^b_{i,k}}(x) := R_i x - (-1)^i (k+1) b_i, \quad x \in T^b_{i,k}.$$

Clearly $\nabla I_{T^a_{i,k}} = \nabla I_{T^b_{i,k}} = R_i$. In particular the strains are independent of k, while the deformations are k-dependent.

Region $\Delta^{a,n}_{1,k}$. We define $I_{\Delta^{a,n}_{1,k}}$ as the interpolation in (8.5), with points

$$p_1 := (r_{n-1,1}, -r_{n-1,1}) + t^k_1, \quad p_2 := (r_{n,1}, r_{n,1}) + t^k_1, \quad p_3 := (r_{n-1,1}, r_{n-1,1}) + t^k_1,$$

and corresponding values

$$v_1 := R_{-\alpha}p_1 + (k+1)\tau\varepsilon b_1, \quad v_j := R_{-\alpha}p_j + k\tau\varepsilon b_1, \; j = 2, 3.$$

The gradient of the interpolation is then

$$\begin{aligned}
\nabla I_{\Delta^{a,n}_{1,k}} &= R_{-\alpha}\begin{pmatrix} r_{n-1,1} \\ -r_{n-1,1}\end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2r_{n-1,1}} \\ -\frac{1}{2r_{n-1,1}}\end{pmatrix} + \tau\varepsilon b_1 \otimes \begin{pmatrix} \frac{1}{2r_{n-1,1}} \\ -\frac{1}{2r_{n-1,1}}\end{pmatrix} \\
&\quad + R_{-\alpha}\begin{pmatrix} r_{n,1} \\ r_{n,1}\end{pmatrix} \otimes \begin{pmatrix} \frac{1}{r_{n,1}-r_{n-1,1}} \\ 0\end{pmatrix} + R_{-\alpha}\begin{pmatrix} r_{n-1,1} \\ r_{n-1,1}\end{pmatrix} \otimes \begin{pmatrix} \frac{-r_{n,1}-r_{n-1,1}}{2r_{n-1,1}(r_{n,1}-r_{n-1,1})} \\ \frac{1}{2r_{n-1,1}}\end{pmatrix} \\
&= R_{-\alpha} + \frac{1}{r_{n,1}}\tau\varepsilon b_1 \otimes \begin{pmatrix} 1 \\ -1\end{pmatrix},
\end{aligned}$$

where we have used that $r_{n-1,1} = r_{n,1}/2$.

Region $\Delta^{b,n}_{1,k}$. The interpolation $I_{\Delta^{b,n}_{1,k}}$ is defined as in (8.5), with points

$$p_1 := (r_{n-1,1}, -r_{n-1,1}) + t^k_1, \quad p_2 := (r_{n,1}, -r_{n,1}) + t^k_1, \quad p_3 := (r_{n,1}, r_{n,1}) + t^k_1,$$

and corresponding values

$$v_j := R_{-\alpha}p_j + (k+1)\tau\varepsilon b_1, \; j = 1, 2, \quad v_3 := R_{-\alpha}p_3 + k\tau\varepsilon b_1.$$

The gradient of the interpolation is

$$
\begin{aligned}
\nabla I_{\Delta_{1,k}^{b,n}} &= R_{-\alpha}\begin{pmatrix} r_{n-1,1} \\ -r_{n-1,1}\end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{r_{n,1}-r_{n-1,1}} \\ 0 \end{pmatrix} + R_{-\alpha}\begin{pmatrix} r_{n,1} \\ -r_{n,1}\end{pmatrix} \otimes \begin{pmatrix} \frac{r_{n,1}+r_{n-1,1}}{2r_{n,1}(r_{n,1}-r_{n-1,1})} \\ -\frac{1}{2r_{n,1}} \end{pmatrix} \\
&\quad + R_{-\alpha}\begin{pmatrix} r_{n,1} \\ r_{n,1}\end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2r_{n,1}} \\ \frac{1}{2r_{n,1}} \end{pmatrix} - \tau\varepsilon b_1 \otimes \begin{pmatrix} \frac{1}{2r_{n,1}} \\ \frac{1}{2r_{n,1}} \end{pmatrix} \\
&= R_{-\alpha} - \frac{1}{2r_{n,1}}\tau\varepsilon b_1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}
$$

Region $\Delta_{2,k}^{a,n}$. We define $I_{\Delta_{2,k}^{a,n}}$ as the interpolation in (8.5), with points

$$
\begin{aligned}
p_1 &:= (-r_{n-1,2}, -r_{n-1,2}) + t_2^k, \\
p_2 &:= (-r_{n,2}, r_{n,2}) + t_2^k, \\
p_3 &:= (-r_{n-1,2}, r_{n-1,2}) + t_2^k,
\end{aligned}
$$

and corresponding values

$$
v_1 := R_\alpha p_1 - (k+1)\tau\varepsilon b_2, \quad v_j := R_\alpha p_j - k\tau\varepsilon b_2,\ j = 2, 3.
$$

The gradient of the interpolation is then

$$
\begin{aligned}
\nabla I_{\Delta_{2,k}^{a,n}} &= R_\alpha\begin{pmatrix} -r_{n-1,2} \\ -r_{n-1,2}\end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{2r_{n-1,2}} \\ -\frac{1}{2r_{n-1,2}} \end{pmatrix} - \tau\varepsilon b_2 \otimes \begin{pmatrix} -\frac{1}{2r_{n-1,2}} \\ -\frac{1}{2r_{n-1,2}} \end{pmatrix} \\
&\quad + R_\alpha\begin{pmatrix} -r_{n,2} \\ r_{n,2}\end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{r_{n,2}-r_{n-1,2}} \\ 0 \end{pmatrix} + R_\alpha\begin{pmatrix} -r_{n-1,2} \\ r_{n-1,2}\end{pmatrix} \otimes \begin{pmatrix} \frac{r_{n,2}+r_{n-1,2}}{2r_{n-1,2}(r_{n,2}-r_{n-1,2})} \\ \frac{1}{2r_{n-1,2}} \end{pmatrix} \\
&= R_\alpha + \frac{1}{r_{n,2}}\tau\varepsilon b_2 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned}
$$

where we have used that $r_{n-1,2} = r_{n,2}/2$.

Region $\Delta_{2,k}^{b,n}$. The interpolation $I_{\Delta_{2,k}^{b,n}}$ is defined as in (8.5), with points

$$
p_1 := (-r_{n-1,2}, -r_{n-1,2}) + t_2^k, \quad p_2 := (-r_{n,2}, -r_{n,2}) + t_2^k, \quad p_3 := (-r_{n,2}, r_{n,2}) + t_2^k,
$$

and corresponding values

$$
v_j := R_\alpha p_j - (k+1)\tau\varepsilon b_2,\ j = 1, 2, \quad v_3 := R_\alpha p_3 - k\tau\varepsilon b_2.
$$

The gradient of the interpolation is

$$\nabla I_{\Delta^{b,n}_{2,k}} = R_\alpha \begin{pmatrix} -r_{n-1,2} \\ -r_{n-1,2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{r_{n,2}-r_{n-1,2}} \\ 0 \end{pmatrix} + R_\alpha \begin{pmatrix} -r_{n,2} \\ -r_{n,2} \end{pmatrix} \otimes \begin{pmatrix} -\frac{r_{n,2}+r_{n-1,2}}{2r_{n,2}(r_{n,2}-r_{n-1,2})} \\ -\frac{1}{2r_{n,2}} \end{pmatrix}$$
$$+ R_\alpha \begin{pmatrix} -r_{n,2} \\ r_{n,2} \end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{2r_{n,2}} \\ \frac{1}{2r_{n,2}} \end{pmatrix} + \tau\varepsilon b_2 \otimes \begin{pmatrix} -\frac{1}{2r_{n,2}} \\ \frac{1}{2r_{n,2}} \end{pmatrix}$$
$$= R_\alpha + \frac{1}{2r_{n,2}} \tau\varepsilon b_2 \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The definition of the piecewise constant strain. We define the strain $\beta : \Omega \to \mathbb{R}^{2\times 2}$ as

$$\beta := \begin{cases} Id & \text{in } (D_i^0 + t_i^k),\ k = 0, \dots, N_i,\ i = 1, 2, \\ \nabla I_{\Delta^{z,n}_{i,k}} & \text{in } \Delta_k^{z,n},\ z \in \{a, b\},\ i = 1, 2,\ k = 0, \dots, N_i,\ n = 1, \dots, \bar{n}_i, \\ R_\alpha & \text{in } \Sigma_\alpha \cup \bigcup_{k=0}^{N_2} (Q^r_{2,k} \cup T^a_{2,k} \cup T^b_{2,k}), \\ R_{-\alpha} & \text{in } \Sigma_{-\alpha} \cup \bigcup_{k=0}^{N_1} (Q^l_{1,k} \cup T^a_{1,k} \cup T^b_{1,k}). \end{cases} \tag{8.7}$$

Remark 8.2.1. A difference between our construction and the ones in [LL16] and [FGS25] is that the separation of the half squares introduced by the Burgers' vector is k-dependent. In other words, while in each square the additional horizontal shift—compared with the one above—is of one Burgers' vector, the total shift in the k-square is the cumulative effect of all the shifts above it, since once a dislocation is added in the square above, the square below will have to accommodate it as well. This is particularly clear observing the illustration of the deformation in Figure 8.3.

8.3 Admissibility and energy of the grain boundary

8.3.1 Admissibility

In this section we check that the strain β is admissible, namely that it satisfies assumptions (H1)–(H3), and identify the set S where the curl concentrate.

By an application of Lemma 6.5.1, it is easy to see that the only lines where the curl of β may concentrate are $\Sigma_1 \cap \Sigma_2$, and ∂D_i^0 (with its translations). Indeed, one can easily check that

$$\nabla I_{\Delta^{a,n}_{i,k}}((-1)^{i+1}e_1 + e_2) = R_i((-1)^{i+1}e_1 + e_2), \qquad i = 1, 2,$$
$$\nabla I_{\Delta^{b,n}_{i,k}}((-1)^{i+1}e_1 - e_2) = R_i((-1)^{i+1}e_1 - e_2), \qquad i = 1, 2,$$

showing that no curl concentrates along the interfaces between $\Delta^{a,n}_{i,k}$ and $T^a_{i,k}$ and between $\Delta^{b,n}_{i,k}$ and $T^a_{i,k}$, for $i = 1, 2$, $k = 0, \dots, N_i$ and $n = 1, \dots, \bar{n}_i$. Moreover, since $I_{\Delta^{a,n}_{i,k}}$ and $I_{\Delta^{b,n}_{i,k}}$ (and, similarly, $I_{\Delta^{b,n}_{i,k}}$ and $I_{\Delta^{a,n+1}_{i,k}}$) have the same values on the common vertices, no curl concentrates on the interface between the corresponding

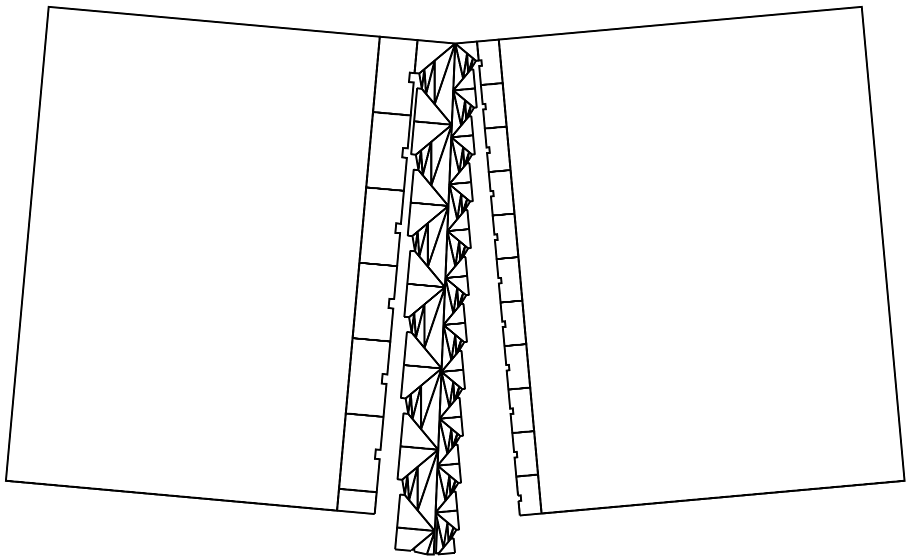

Figure 8.3: The deformed rectangle with $\eta := -\pi/3$ and $\theta := \pi/6$.

regions. We recall that the curl energy penalizes the length of the interfaces between constant values of the strain that are not rank-one connected along the direction of the interface. While the length of ∂D_0 is small, and hence a curl concentration there would be energetically affordable in principle, the length of $\Sigma_1 \cap \Sigma_2$ is of order one and hence we need to ensure the values of β across it are rank-one connected. Here is where we fix the length of the square Q_i, namely $2r_{\bar{n}_i}$.

By imposing that the constant values of β in $\Delta^{b,\bar{n}_1}_{1,k}$ and $\Delta^{b,\bar{n}_2}_{2,k}$ are rank-one connected along the common boundary we get the condition

$$\left[\left(R_{-\alpha} - \frac{1}{2r_{\bar{n}_1}}\tau\varepsilon b_1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) - \left(R_{\alpha} - \frac{1}{2r_{\bar{n}_2}}\tau\varepsilon b_2 \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right] e_2 = 0,$$

which simplifies to

$$\frac{1}{r_{\bar{n}_1}}\tau\varepsilon b_1 + \frac{1}{r_{\bar{n}_2}}\tau\varepsilon b_2 = 4\sin\alpha e_1. \tag{8.8}$$

By using the explicit forms of b_1 and b_2, (8.8) fixes the free parameters $r_{\bar{n}_i}$. Indeed, we have

$$r_{\bar{n}_1} = -\frac{\tau\varepsilon}{4}\frac{\sin(\eta-\theta)}{\sin\alpha\sin\theta}, \qquad r_{\bar{n}_2} = \frac{\tau\varepsilon}{4}\frac{\sin(\eta-\theta)}{\sin\alpha\sin\eta}. \tag{8.9}$$

Note that both $r_{\bar{n}_1}$ and $r_{\bar{n}_2}$ needs to be positive and less than $L/2$. While the second condition is clearly satisfied for $\varepsilon \ll 1$, the first amounts to have

$$\begin{aligned} &\sin(\eta)\tan^{-1}(\theta) - \cos(\eta) < 0, \\ &\cos(\theta) - \sin(\theta)\tan^{-1}(\eta) > 0. \end{aligned} \tag{8.10}$$

Without loss of generality, we can assume to have $\sin(\eta) > 0$ and $\sin(\theta) < 0$. Indeed, if this is not the case we can swap η with $\eta + \pi$ or θ with $\theta + \pi$ without changing the Bravais' lattice $\mathcal{B}$. Then, (8.10) reduce to

$$\tan^{-1}(\theta) < \tan^{-1}(\eta),$$

that is satisfied up to swapping b_1 and b_2.

Under the additional condition (8.8) the strain β defined in (8.7) satisfies (H1), with $S := \cup_{i,k}(D_i^0 + t_i^k)$, and (H3). We now check condition (H2). To compute the circulation of β on ∂D_i^0 it is sufficient to take γ as a closed curve surrounding $D_i^0 + t_i^k$ (see Section 6.5). With no loss of generality we can take a concentric square, with side-length $2r_{n,i}$. Since $\nabla I_{\Delta_{i,k}^{a,n}}$ and $\nabla I_{\Delta_{i,k}^{b,n}}$ (for $i = 1, 2$) are rank-one connected on their common boundary, we can take either value in the computation of the circulation of β on γ. One gets, for $i = 1$,

$$\int_\gamma \beta\, t ds = 2r_{n,1}\left(\nabla I_{\Delta_1^{b,n}} e_2 - R_{-\alpha} e_2\right) = -\tau\varepsilon b_1,$$

and for $i = 2$,

$$\int_\gamma \beta\, t ds = 2r_{n,2}\left(R_\alpha e_2 - \nabla I_{\Delta_2^{b,n}} e_2\right) = -\tau\varepsilon b_2.$$

Hence, condition (H2) is satisfied, and the strain β is admissible.

Remark 8.3.1 (The square lattice)**.** We note that the distance $r_{\bar{n}_i}$ between dislocations with Burgers vector b_i in (8.9) is in complete agreement with the computations of Read and Shockley. Indeed, in the special case of the square lattice, we have $|\sin(\eta - \theta)| = 1$; moreover, by fixing e.g. $\eta \in (0, \pi/2)$, we have that $\theta = \eta + \frac{3}{2}\pi$, hence the spacings of the dislocations are

$$r_{\bar{n}_1} \sim \frac{\tau\varepsilon}{\sin\alpha}\frac{1}{\cos\eta}, \qquad r_{\bar{n}_2} \sim \frac{\tau\varepsilon}{\sin\alpha}\frac{1}{\sin\eta},$$

exactly as computed by Read and Shockley.

Remark 8.3.2 (The case of symmetric grain boundaries)**.** Suppose that $\eta = 0$, namely $b_1 = e_1$. The construction presented above does not immediately work in this case, since in (8.9) the spacing of the dislocations with Burgers vector b_2 is infinite. This is due to the fact that to achieve symmetric boundary conditions only e_1 is needed.

To treat this special case one has to make the simple adaptation of taking $b_2 = e_1$ in the construction, and consequently $r_{\bar{n}_1} = r_{\bar{n}_2}$. Then (8.8) becomes

$$\frac{2}{r_{\bar{n}_1}}\tau\varepsilon e_1 = 4\sin\alpha e_1,$$

namely

$$r_{\bar{n}_1} = \frac{\tau\varepsilon}{2\sin\alpha}.$$

8.3.2 Energy

We compute the elastic and curl energy on a single square Q_i^k, for $i = 1, 2$, and then multiply it by the number of the squares in each vertical strip, since the constant values of β are k-independent.

The elastic energy

For the elastic energy, note that by (8.2), for $\varepsilon \ll 1$ we have

$$\begin{aligned}\int_{\Omega\setminus B_{\lambda\varepsilon}(S))} \mathcal{W}(\beta)\,dx &\le C\int_\Omega \operatorname{dist}^2(\beta,\mathrm{SO}(2))dx\\ &\le C\sum_{i=1}^{2}\sum_{k=0}^{N_i}\sum_{n=1}^{\bar n_i}\int_{\Delta^{a,n}_{i,k}\cup\Delta^{b,n}_{i,k}} \operatorname{dist}^2(\beta,\mathrm{SO}(2))dx,\end{aligned}$$

where $N_i+1=\lceil L/r_{\bar n_i}\rceil$ is the number of squares Q_i^k in the vertical strip Σ_i. Since

$$\operatorname{dist}^2(\beta,\mathrm{SO}(2)) \le C\frac{1}{r^2_{n,i}}\varepsilon^2 \quad \text{in } \Delta^{a,n}_{i,k}\cup\Delta^{b,n}_{i,k},\ i=1,2,$$

with $C>0$, we then have the estimate

$$\begin{aligned}\int_\Omega \operatorname{dist}^2(\beta,\mathrm{SO}(2))dx &\le C\varepsilon^2\sum_{i=1}^{2}\sum_{k=0}^{N_i}\sum_{n=1}^{\bar n_i}\frac{1}{r^2_{n.i}}\mathcal{L}^2(D_i^n)\\ &= C\varepsilon^2\sum_{i=1}^{2}\sum_{k=0}^{N_i}\sum_{n=1}^{\bar n_i}\frac{1}{r^2_{n,i}}(r^2_{n,i}-r^2_{n-1,i})\\ &\le C\varepsilon^2\sum_{i=1}^{2}(N_i+1)\bar n_i \le C\varepsilon^2\sum_{i=1}^{2}\frac{\bar n_i}{r_{\bar n_i,i}}.\end{aligned}$$

We fix now $\bar n_i := \lceil|\log(\alpha)|\rceil$. Since $\alpha\sim\sin(\alpha)$ for α small, by (8.9) we conclude that

$$\int_\Omega \operatorname{dist}^2(\beta,\mathrm{SO}(2))dx \le C\varepsilon\alpha|\log\alpha|.$$

By the definition of the energy (8.1), the elastic energy contribution of β is of order $\alpha|\log\alpha|$.

The core energy

The conditions we imposed on the strain β in Section 8.3.1 ensure that the curl of β is concentrated on the boundaries of the translated inner squares ∂D_i^0 (within Q_i^k). Recall that, since $r_{\bar n_i,i}\sim\varepsilon/\alpha$, $\bar n_i=\lceil|\log(\alpha)|\rceil$, and $2^{\bar n_i}r_{0,i}=r_{\bar n_i,i}$, we get $r_{0,i}\sim\varepsilon$. The corresponding energy contribution is then, from (8.1), of order

$$\frac{\alpha}{\varepsilon}\frac{1}{\varepsilon}\mathcal{L}^2(B_{\lambda\varepsilon}(\partial D_i^0))\sim\alpha,$$

since the number of squares Q_i^k in the vertical strip Σ_i is of order α/ε. This contribution is smaller than the elastic energy, which is the dominant term in the energy.

The total energy, from (8.1), can be estimated from above as

$$E_\varepsilon(\beta,S)\le C\alpha(|\log\alpha|+1),$$

where $S=\cup_{i,k}(D_i^0+t_i^k)$, in agreement with the computations of Read and Shockley.

Remark 8.3.3. Notice that the choice $\bar{n}_i = \lceil |\log(\alpha)| \rceil$ is energically optimal once $r_{\bar{n}_i,i}$ is taken of order ε/α. Indeed, leaving $\bar{n}_i$ as free parameters, the energy would be bounded by

$$C \sum_{i=1}^{2} \left(\alpha \bar{n}_i + \frac{1}{2^{\bar{n}_i}} \right).$$

It is matter of a simple computation to show that the map

$$x \mapsto \alpha x + \frac{1}{2^x}$$

is minimized at $x \sim -\log(\alpha)$.

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