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On a strain gradient plasticity model accounting for the Burgers vector and plastic spin

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Contents

Abstract

In this thesis we prove the existence of solutions, in the rate-independent case, to a strain gradient elasto-plasticity model for small deformations introduced by Gurtin. The key features of this model are the dependence on the Burgers vector, which admits non trivial plastic spin, and on the gradient of the plastic strain. We apply the energetic method by Mielke and coauthors, which is a well-established approach for proving existence of quasistatic evolutions. We show that this notion of weak solution is strong enough to give a mathematically sound meaning to the constitutive equations and to the flow rule of the model. Finally, by an asymptotic analysis we investigate the relation of the Gurtin model with the simplified model for plastically irrotational bodies by Gurtin and Anand and with the classical Prandtl-Reuss model of perfect plasticity.

Chapter 1

Introduction

Plasticity is the property of a material to sustain permanent deformations. Plastic behavior is observed in most materials, however, its underlying mechanism can vary widely. In this work we will focus on plasticity in metals. A classical model is that of Prandtl and Reuss of perfect plasticity. This theory applies to ideal metals that do not show hardening or softening behavior. Moreover, it is a small strain model, i.e., it is assumed that the deformation of the body is small relatively to its size. In such theories the strain tensor, which usually depends in a non linear way on the gradient of the displacement u, can be linearized. More precisely, the strain tensor is approximated by the symmetric gradient of the displacement function u

$$Eu = \frac{1}{2}(\nabla u + \nabla u^T)\,.$$

The strain is then additively decomposed to highlight the elastic and plastic parts

$$Eu = H^e + H^p\,.$$

The fundamental hypothesis of the Prandtl-Reuss model is that the response of the body is ideally elastic until a critical threshold is reached; then plastic deformation occurs without any further increase of the stress. Mathematically, this is modeled by introducing a convex and closed region K in the stress space, containing the origin as an interior point. The boundary of K, called the yield surface, represents the activation threshold for plastic behavior. The Cauchy's stress T, which depends linearly on the elastic strain only, satisfies the classical Hooke's law $T = \mathbb{C}H^e$. Its deviatoric part T_D is constrained to belong to the region K. Finally, the Prandtl-Reuss flow rule states that

$$\dot{H}^p \in N_K(T_D)\,, \tag{1.1}$$

where $\dot{H}^p$ denotes the time derivative of H^p and $N_K(T_D)$ is the normal cone to K at the point T_D. Equation (1.1) ensures that plastic behavior develops only when the stress T_D is at the boundary of the region K. Different choices for the region K produce different flow rules (e.g in the case of a ball centered at the origin one retrieves the Von Mises flow rule).

The existence of solutions to the Prandtl-Reuss model was first proved by Suquet in [27]. Another proof of existence was given in [15] by Dal Maso, DeSimone, and Mora, using the energetic approach developed by Mielke et al. (see [17]). Perfect plasticity is, however, mainly a conceptual tool and usually does not provide quantitatively accurate predictions. This fact is due to different reasons, some of which we try to highlight here.

First of all, experimental evidence suggests that plasticity depends on the material size, i.e., the smaller the length scale of the crystal lattice (with respect to the body length scale), the stronger the material in its plastic response, see [5]. These effects cannot be captured by the Prandtl-Reuss model since it is invariant to changes in the length scale. One way to include them is to introduce size-dependence via strain gradient models.

Secondly, it is well known that plasticity in metals is mainly caused at the microscopic level by defects of the crystal lattice, called dislocations. The creation, movement, and annihilation of such defects are the main mechanisms for the plastic response of the body. However, it is not clear how this microscopic description is related to the Prandtl-Reuss theory at the continuum level. Furthermore, dislocations pile-up in some regions of the body is responsible for the hardening phenomenon, which is neglected in perfect plasticity. Several continuum models have been proposed to take into account some dependence on the dislocations density. This is usually done by including in the model the so-called macroscopic Burgers vector B, that is, the curl of the plastic strain

$$B := \mathrm{curl}(H^p)\,.$$

Indeed, from a mechanical viewpoint, B quantifies macroscopically the amount and the location of the defects.

In this thesis we study a strain gradient model accounting for the macroscopic Burgers vector, introduced by Gurtin in [10]. Here, the starting assumption is the decomposition of the whole gradient ∇u in an elastic and a plastic part, which we will improperly call elastic and plastic strain

$$\nabla u = H^e + H^p\,.$$

Indeed, to be precise, only the symmetric parts contribute to the strain, while H^e_{skew} and H^p_{skew} are usually called elastic and plastic spin. The Burgers vector enters in the model through the free energy of the body

$$\frac{1}{2}\int_\Omega \mathbb{C}H^e_{sym} : H^e_{sym}\; dx + \frac{\mu L^2}{2}\int_\Omega |B|^2\; dx\,, \tag{1.2}$$

where $L > 0$ is a length scale associated with B and $\mu > 0$ is a material constant. It is important to note that the Burgers vector depends on the plastic spin

$$B = \mathrm{curl}(H^p) = \mathrm{curl}(H^p_{sym}) + \mathrm{curl}(H^p_{skew})\,.$$

The flow rule is a generalization of that of Von Mises and takes the following form:

$$(\dot{H}^p, \nabla \dot{H}^p_{sym}) \in N_{\mathcal{K}}(T^p, K_{diss})\,, \tag{1.3}$$

where T^p and K_{diss} are suitably defined plastic stress tensors that satisfy the equation

$$T_D - \operatorname{curl}(\mu L B^T)_D = T^p - \operatorname{div} K_{diss}\,. \tag{1.4}$$

Here $\mathcal{K}$ is a closed and convex set that depends on two scale length, χ and h, associated to the plastic spin and the strain gradient, respectively. Note that the flow rule, which is still rate-independent, depends on the strain gradient ∇H^p_{sym}. Taking the limit as $L, \chi, h \to 0$, one formally retrieves the classical Von Mises flow rule. For a more detailed description of the model we refer to Chapter 3 and to the original work by Gurtin [10].

The main objective of this thesis is to show the existence of solutions to the Gurtin model. Inspired by the work by Dal Maso, DeSimone, and Mora we use the same energetic approach and we introduce the concept of energetic solution. This is a weak notion of solution where no a priori time regularity is required. An energetic solution has to satisfy two conditions: a global stability, ensuring that the body is at the equilibrium at every time during the evolution, and an energy balance, stating that the sum of the stored energy and the plastic dissipation equals the work done by the external forces. This framework for quasistatic evolution problems was developed by Mielke and coauthors in [16, 17, 19, 20, 21]. The strategy to prove the existence of solutions consists in discretizing the time interval $[0, T]$ in which the evolution occurs. An incremental minimization problem is then solved at each discrete time, using the direct method of the Calculus of Variations. Since the flow rule (1.3) is rate-independent, the plastic dissipation has linear growth, hence it is natural to assume H^p_{sym} to be a function with bounded variation. Similarly, by the expression of the free energy (1.2), H^e_{sym} and B are assumed to be L^2 functions. Moreover, in order to guarantee compactness of minimizing sequences, a so-called safe-load condition is needed as in the case of perfect plasticity. Approximated evolutions are then constructed as piecewise-constant-in-time interpolants. Passing to the limit in the discretization parameter, we finally show that, up to extracting a subsequence, these approximate solutions converge to an energetic solution to the Gurtin model.

Subsequently, we show that the notion of energetic solution is strong enough to recover in a suitable sense the equilibrium equations and the flow rule (1.3)-(1.4). We first prove that energetic solutions are absolutely continuous in time, therefore we can speak of their time derivatives. We then show that the flow rule holds in an integral form, and a pointwise version can be deduced at Lebesgue points. Moreover, since H^p_{sym} is a function of bounded variation and in particular ∇H^p_{sym} is only a measure, an additional equation involving the singular part of ∇H^p_{sym} is needed to supplement the flow rule. As a corollary, we show that the evolution of the elastic strain and the Burgers vector is completely determined by the initial configuration of the body, the applied forces, and the boundary conditions.

Finally, we study the asymptotic behavior of energetic solutions with respect to the parameters χ, h, and L. We show that, as $\chi \to \infty$, it is possible to extract a subsequence converging in a suitable sense to a solution to the Gurtin-Anand model. The latter, studied in [6] by Giacomini and Lussardi, is a simplification of the Gurtin model for plastically irrotational bodies, i.e., satisfying $H^p_{skew} = 0$. In particular, in this model the Burgers vector depends only on the (proper) plastic strain H^p_{sym}. To obtain this convergence result it is crucial to refine the estimates providing the absolute continuity in time of solutions and to make explicit their dependence on χ. Lastly, we show that, as $h, L \to 0$, energetic solutions to the Gurtin model converge in a suitable sense, up to subsequences, to an energetic solution to the Prandtl-Reuss model in the sense of Dal Maso, DeSimone, and Mora. This result holds independently of the behavior of χ. We note that this asymptotic analysis cannot be deduced by directly applying the result by Mielke, Roubíček, and Stefanelli in [18] about the convergence of quasistatic evolutions. This is due to the possible concentration on the boundary of the plastic strain, that is just a measure in the energetic formulation of the Prandtl-Reuss model. To overcome this difficulty we rely again on the safe-load condition.

To conclude we mention a possible extension of this work. The Gurtin model can be described as phenomenological, meaning that it is motivated by experimental considerations. Recently, strain gradient models have been rigorously derived starting from discrete models of dislocations, see [1, 4, 7, 8, 22, 23, 26]. For instance, in [13] Garroni, Leoni, and Ponsiglione deduce a stationary model, where the free energy depends, as in the Gurtin model, on the Burgers vector. However, the dependence is not quadratic and the free energy has the following form:

$$\frac{1}{2}\int_\Omega \mathbb{C}H^e_{sym} : H^e_{sym}\ dx + \frac{\mu L^2}{2}\int_\Omega \psi(B)\ dx\,,$$

where ψ is positively 1-homogeneous. It would be interesting to see if the results of this thesis could be adapted to this model. We believe that it should still be possible to prove the existence of a solution in the energetic sense, with some minor changes. However, it is not clear what form the flow rule should have and whether the notion of energetic solution is strong enough to give a mathematically sound meaning to it.

The thesis is organized as follows. In Chapter 2 we introduce the notation and we give a few preliminary results. In Chapter 3 we describe the Gurtin model and we specify the functional setting. In Chapter 4 we prove the existence of energetic solutions to the Gurtin model. In Chapter 5 we show how to give a meaning to the constitutive equations and the flow rule. In Chapter 6 we study the asymptotics of energetic solutions with respect to the parameters χ, h, and L.

Chapter 2

Notation and preliminaries

2.1 Notation

Matrices

We denote by $M^{n\times m}$ the space of real $n \times m$ matrices. We will work for the entirety of this paper in the case $n = m = 3$. $M^{3\times 3}$ is naturally endowed with the Frobenius norm, denoted by $|A|$, induced by the inner product

$$A : B = \sum_{i,j=1}^{3} a_{ij} b_{ij} \,, \tag{2.1}$$

where $A = (a_{ij})$ and $B = (b_{ij})$ belong to $M^{3\times 3}$. We denote by $M^{3\times 3}_{sym}$ and $M^{3\times 3}_{skew}$ the subspaces of $M^{3\times 3}$ given by symmetric and skew-symmetric matrices, respectively. It is well known that $M^{3\times 3}_{sym}$ is the orthogonal complement of $M^{3\times 3}_{skew}$ with respect to the inner product (2.1). We denote by $M^{3\times 3}_{D}$ the subspace of $M^{3\times 3}$ given by all the matrices with vanishing trace, that is,

$$\mathrm{Tr}(A) = \sum_{i=1}^{3} a_{ii} = 0 \,.$$

Such matrices are called deviatoric. To simplify the notation we write $M^{3\times 3}_{D,sym}$ in place of $M^{3\times 3}_{sym} \cap M^{3\times 3}_{D}$. Given a matrix $A \in M^{3\times 3}$ we denote the projection of A on the subspaces of symmetric, skew-symmetric, and deviatoric matrices by A_{sym}, A_{skew}, and A_D, respectively:

$$\begin{aligned} A_{sym} &= \frac{1}{2}(A + A^T) \,, \\ A_{skew} &= \frac{1}{2}(A - A^T) \,, \\ A_D &= A - \frac{1}{3}\,\mathrm{Tr}(A)\,\mathrm{Id} \,. \end{aligned}$$

Here Id denotes the identity matrix in $M^{3\times 3}$. For a function $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ we write $E\varphi$ in place of $(\nabla\varphi)_{sym}$ to denote the projection of the gradient on the subspace of symmetric matrices. $E\varphi$ is called symmetric gradient.

We denote by $M^{3\times3\times3}$ the space of third order tensors endowed with the inner product

$$A : B = \sum_{i,j,k=1}^{3} a_{ijk} b_{ijk}\,,$$

where $A = (a_{ijk})$ and $B = (b_{ijk})$ belong to $M^{3\times3\times3}$. The induced norm is denoted by $|\cdot|$. We use the notation $M^{3\times3\times3}_{sym}$, $M^{3\times3\times3}_{skew}$, and $M^{3\times3\times3}_{D}$ to indicate the subspaces of $M^{3\times3\times3}$ given by all symmetric, skew-symmetric, and deviatoric tensors in the first two subscripts, respectively. More precisely, a tensor $A = (a_{ijk})$ is said to be symmetric, skew-symmetric or deviatoric in the first two subscripts if, respectively,

$$\begin{aligned} a_{ijk} &= a_{jik} && \forall\ i,j,k = 1,2,3\,, \\ a_{ijk} &= -a_{jik} && \forall\ i,j,k = 1,2,3\,, \\ \sum_{i=1}^{3} a_{iij} &= 0 && \forall\ j = 1,2,3\,. \end{aligned}$$

Given a tensor $A = (a_{ijk})$ we denote the projection of A on the subspaces of symmetric, skew-symmetric, and deviatoric tensors in the first two subscripts by A_{sym}, A_{skew}, and A_D, respectively:

$$\begin{aligned} (A_{sym})_{ijk} &= \frac{1}{2}(a_{ijk} + a_{jik})\,, \\ (A_{skew})_{ijk} &= \frac{1}{2}(a_{ijk} - a_{jik})\,, \\ (A_D)_{ijk} &= a_{ijk} - \delta_{ij}\frac{1}{3}\sum_{h=1}^{3} a_{hhk}\,. \end{aligned}$$

where δ_{ij} is the Kronecker delta.

For an $M^{3\times3}$-valued function $\Phi : \mathbb{R}^3 \to M^{3\times3}$ we define

$$(\nabla\Phi)_{ijk} = \frac{\partial \Phi_{ij}}{\partial x_k}\,, \quad (\operatorname{div}\Phi)_i = \sum_{j=1}^{3} \frac{\partial \Phi_{ij}}{\partial x_j}\,, \quad (\operatorname{curl}\Phi)_{ij} = \sum_{p,q=1}^{3} \epsilon_{ipq} \frac{\partial \Phi_{jq}}{\partial x_p}\,,$$

where ϵ_{ijk} is the Levi-Civita symbol. Similarly, for an $M^{3\times3\times3}$-valued function $\Phi : \mathbb{R}^3 \to M^{3\times3\times3}$ we define

$$(\operatorname{div}\Phi)_{ij} = \sum_{k=1}^{3} \frac{\partial \Phi_{ijk}}{\partial x_k}\,.$$

Given two vectors $a, b \in \mathbb{R}^3$ we denote by $a \otimes b$ the matrix defined by $(a \otimes b)_{ij} = a_i b_j$ and we write $a \odot b$ in place of $(a \otimes b)_{sym}$.

Functional spaces

We denote by $L^p(U;X)$ the space of p-summable functions defined on an open and Lipschitz subset $U \subset \mathbb{R}^n$ with values in a Banach space X. When $1 < p < +\infty$ the dual of $L^p(U;X)$ can be identified with $L^q(U;X')$, where q is the conjugate exponent of p. In particular, this is true whenever X has finite dimension (e.g. $X = \mathbb{R}^n$ or $X = M^{3\times 3}$); in this case $X' = X$.

When X is finite dimensional we denote by $W^{1,p}(U;X)$ the usual Sobolev space given by all functions in $L^p(U;X)$ with distributional gradient in $L^p(U;X)$. When $p = 2$ we write $H^1(U;X)$ in place of $W^{1,2}(U;X)$.

By the Rellich-Kondrachov theorem the space $W^{1,p}(U;X)$ is compactly embedded in $L^p(U;X)$.

Every Sobolev function $f \in W^{1,p}(U;X)$ admits a trace, which we denote by $\gamma(f)$ or f when it is clear from the context, as an element of $L^p(\partial U;X)$. The trace operator

$$\gamma : W^{1,p}(U;X) \to L^p(\partial U;X)$$

is continuous and compact. The image of the trace operator is the space $W^{1-\frac{1}{p},p}(\partial U;X)$ that we denote by $H^{\frac{1}{2}}(\partial U;X)$ when $p = 2$. The kernel of γ is denoted by $W^{1,p}_0(U;X)$. The dual space of $W^{1,p}(U;X)$ is denoted by $W^{-1,q}(U;X)$ where q is the conjugate exponent of p. The same notation is used for the image of the trace operator, so that $(H^{\frac{1}{2}}(\partial U;X))' = H^{-\frac{1}{2}}(\partial U;X)$.

For $X = M^{3\times 3}$ or $X = M^{3\times 3\times 3}$ we denote by $L^p_{\mathrm{div}}(U;X)$ the space of all functions in $L^p(U;X)$ with distributional divergence in $L^p(U;Y)$. Here Y is the correct space in which the divergence takes values (i.e. for $X = M^{3\times 3}$ it is $Y = \mathbb{R}^3$ and for $X = M^{3\times 3\times 3}$ it is $Y = M^{3\times 3}$). For every function $f \in L^p_{\mathrm{div}}(U;X)$ the normal trace $\gamma_\nu(f)$ is defined as an element of $W^{-1+\frac{1}{p},q}(\partial U;Y)$, where q is the conjugate exponent of p. More precisely, $\gamma_\nu(f)$ is defined as follows:

$$\langle \gamma_\nu(f), v\rangle = \int_U (\operatorname{div} f \cdot v + f \cdot \nabla v)\; dx\,,$$

where the dual product on the left-hand side is the one between $W^{-1+\frac{1}{p},q}(\partial U;Y)$ and $W^{1-\frac{1}{p},p}(\partial U;Y)$.

Absolutely continuous functions

We denote by $\mathrm{AC}(0,T;X)$ the space of absolutely continuous functions on $[0,T]$ with values in a Banach space X. We refer to [3] for the main properties of this space. If X is reflexive, every $f \in \mathrm{AC}(0,T;X)$ has a weak time-derivative $\dot f \in L^1(0,T;X)$. Moreover, the fundamental theorem of calculus holds. Namely,

$$f(t) - f(s) = \int_s^t \dot f(\tau)\; d\tau \quad \forall\; s,t \in [0,T]\,.$$

If X is the dual of a separable Banach space, by Theorem 7.1 in [15] every $f \in \mathrm{AC}(0,T;X)$ admits a time-derivative as the weak*-limit of the difference quotients for almost every $t \in [0,T]$. In this case it is possible to show that the map

$$t \mapsto \|\dot{f}(t)\|_X$$

is measurable and belongs to $L^1(0,T;\mathbb{R})$.

We will often use the following generalization of the Arzelà-Ascoli Theorem.

Theorem 2.1.1. *Let X be a reflexive Banach space (resp. the dual of a separable Banach space). Let $(f_k)_{k\in\mathbb{N}} \subset \mathrm{AC}(0,T;X)$ be a sequence of equiabsolutely continuous functions with respect to k such that $f_k(0)$ is bounded in X. Then, there exist a subsequence (f_{k_h}) and a function $f \in \mathrm{AC}(0,T;X)$ such that $f_{k_h}(t) \rightharpoonup f(t)$ (resp. $f_{k_h}(t) \overset{*}{\rightharpoonup} f(t)$) for every $t \in [0,T]$.*

Proof. We give the proof for X a reflexive Banach space. The same argument with minor changes applies to X the dual of a separable Banach space.

By the hypothesis the sequence $(f_k(t))_k$ is uniformly bounded in X for every $t \in [0,T]$. Let $(t_i)_i$ be an enumeration of the rational numbers in the interval $[0,T]$. By the reflexivity of X and a diagonal argument there exist a subsequence $(k_h)_h$ such that

$$f_{k_h}(t_i) \rightharpoonup f(t_i) \quad \text{in } X \quad \forall\, i \in \mathbb{N}\,,$$

for some $f(t_i) \in X$. Let $t \in [0,T]$ and $\varepsilon > 0$. By the hypothesis there exist a constant C and a function $g \in L^1(0,T)$, both independent of k, such that

$$\|f_{k_h}(t_1) - f_{k_h}(t_2)\|_X \leq C \int_{t_1}^{t_2} g(t)\ dt \quad \forall\, h \in \mathbb{N} \tag{2.2}$$

for every $t_1 < t_2 \in [0,T]$. By density there exists $i \in \mathbb{N}$ such that $|t - t_i| \leq \varepsilon$ and we can suppose without loss of generality that $t_i < t$. Then, for every $\psi \in X'$ and for every $l \in \mathbb{N}$ we have

$$\begin{aligned}
&|\langle \psi, f_{k_{h+l}}(t) - f_{k_h}(t)\rangle| \leq |\langle \psi, f_{k_{h+l}}(t) - f_{k_{h+l}}(t_i)\rangle| \\
&+ |\langle \psi, f_{k_{h+l}}(t_i) - f_{k_h}(t_i)\rangle| + |\langle \psi, f_{k_h}(t_i) - f_{k_h}(t)\rangle| \\
&\leq |\langle \psi, f_{k_{h+l}}(t_i) - f_{k_h}(t_i)\rangle| + \|\psi\|_{X'} \|f_{k_{h+l}}(t) - f_{k_{h+l}}(t_i))\|_X \\
&+ \|\psi\|_{X'} \|f_{k_h}(t_i) - f_{k_h}(t))\|_X \leq |\langle \psi, f_{k_{h+l}}(t_i) - f_{k_h}(t_i)\rangle| \\
&+ 2C\|\psi\|_{X'} \int_{t_i}^{t} g(t)\ dt \leq |\langle \psi, f_{k_{h+l}}(t_i) - f_{k_h}(t_i)\rangle| + 2\varepsilon C \|\psi\|_{X'} \|g\|_{L^1}
\end{aligned}$$

Therefore, the sequence $(\langle \psi, f_{k_h}(t)\rangle)_h$ is a Cauchy sequence for every $\psi \in X'$. Let

$$l_\psi = \lim_{h\to\infty} \langle \psi, f_{k_h}(t)\rangle\,.$$

By the Banach-Steinhaus theorem the map

$$\Psi : X' \to \mathbb{R} \ : \ \psi \mapsto l_\psi$$

belongs to X'', hence, by the reflexivity of X we can identify Ψ with an element $f(t) \in X$. In particular we have shown that $f_{k_h}(t) \rightharpoonup f(t)$ in X. We are left to prove that the map

$$f : [0, T] \to X \ : \ t \mapsto f(t)$$

is absolutely continuous. By the weak lower semicontinuity of the norm and (2.2) we deduce that

$$\|f(t_1) - f(t_2)\|_X \leq C \int_{t_1}^{t_2} g(t) \, dt$$

for every $t_1 < t_2 \in [0, T]$. Hence, f belongs to $\mathrm{AC}(0, T; X)$. ■

Measures

We denote by $\mathscr{L}^n$ the n-dimensional Lebesgue measure and by $\mathscr{H}^n$ the n-dimensional Hausdorff measure.

We denote by $\mathscr{M}_b(U; \mathbb{R}^m)$ the space of bounded Borel measures defined on an open and Lipschitz subset $U \subset \mathbb{R}^n$ with values in $\mathbb{R}^m$. $\mathscr{M}_b(U; \mathbb{R}^m)$ is a Banach space with respect to the norm $\|\mu\|_{\mathscr{M}_b} := |\mu|(U)$, where $|\mu|$ is the total variation of μ.

Every measure μ admits a Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is absolutely continuous and μ^s is singular with respect to $\mathscr{L}^n$. If a measure μ is absolutely continuous with respect to $\mathscr{L}^n$, then we will always identify it with its density. In this way we embed continuously $L^1(U; \mathbb{R}^m)$ into $\mathscr{M}_b(U; \mathbb{R}^m)$.

The space $\mathscr{M}_b(U; \mathbb{R}^m)$ is the dual of $C_0(U; \mathbb{R}^m)$, the space of continuous functions $\varphi : U \to \mathbb{R}^m$ such that for every $\varepsilon > 0$ the set

$$\{x \in U \ : \ |\varphi(x)| > \varepsilon\}$$

is compact (see Theorem 6.19 in [25]). Therefore, $\mathscr{M}_b(U; \mathbb{R}^m)$ is naturally endowed with the weak* topology induced by this duality.

Functions with bounded variation

Let $U \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. We denote by $\mathrm{BV}(U; \mathbb{R}^m)$ the space of functions in $L^1(U; \mathbb{R}^m)$ with distributional gradient in $\mathscr{M}_b(U; M^{m \times n})$. $\mathrm{BV}(U; \mathbb{R}^m)$ is a Banach space with the norm

$$\|\psi\|_{\mathrm{BV}} := \|\psi\|_{L^1} + \|\nabla\psi\|_{\mathscr{M}_b} .$$

For a given $\psi \in \mathrm{BV}(U; \mathbb{R}^m)$ we write the Lebesgue decomposition of $\nabla\psi$ as follows:

$$\nabla\psi = \nabla^a\psi + \nabla^s\psi .$$

The space $\mathrm{BV}(U; \mathbb{R}^m)$ is continuously embedded in $L^{\frac{n}{n-1}}(U; \mathbb{R}^m)$.

It is known that $\mathrm{BV}(U; \mathbb{R}^m)$ is the dual space of a separable Banach space (Proposition 2.4 in [24]). We will equip $\mathrm{BV}(U; \mathbb{R}^m)$ with the weak* topology given by this duality.

We denote by $\mathrm{BV}(0, T; X)$ the space of all functions $f : [0, T] \to X$ with bounded variation in $[0, T]$. The variation of a function $f : [0, T] \to X$ is defined as follows:

$$\mathcal{V}(f;0,T) = \sup\left\{\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|_X \;:\; n\in\mathbb{N} \quad \text{and} \right.$$
$$\left. 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T \right\}.$$

Every function $f \in \mathrm{BV}(0,T;X)$ has at most countable many discontinuity points.

We will often use the following generalization of the classical Helly Theorem.

Theorem 2.1.2. *Let X be a reflexive separable Banach space (resp. the dual of a separable Banach space). Let $(f_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(0,T;X)$ be a sequence of functions such that $f_k(0)$ and $\mathcal{V}(f_k;0,T)$ are bounded uniformly with respect to k. Then, there exist a subsequence (f_{k_h}) and a function $f \in \mathrm{BV}(0,T;X)$ such that $f_{k_h}(t) \rightharpoonup f(t)$ (resp. $f_{k_h}(t) \overset{*}{\rightharpoonup} f(t)$) for every $t \in [0,T]$.*

For a proof of this result we refer to [15] (Lemma 7.2) and [2] (Theorem 1.126).

Function with bounded deformation

Let $U \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. We denote by $\mathrm{BD}(U)$ the space of functions with bounded deformation. Precisely, a function f belongs to $\mathrm{BD}(U)$ if it is in $L^1(U;\mathbb{R}^n)$ and its symmetric gradient Ef is a bounded Borel measure. We refer to [28] for a detailed description of such space.

$\mathrm{BD}(U)$ is a Banach space with the norm

$$\|f\|_{\mathrm{BD}} := \|f\|_{L^1} + \|Ef\|_{\mathscr{M}_b}.$$

It is possible to prove that $\mathrm{BD}(U)$ is the dual of a separable Banach space. Therefore, we can equip $\mathrm{BD}(U)$ with the weak* topology given by this duality.

The space $\mathrm{BD}(U)$ is continuously embedded in $L^{\frac{n}{n-1}}(U;\mathbb{R}^n)$.

For every function $f \in \mathrm{BD}(U)$ it is possible to define the trace of f as an element of $L^1(\partial U;\mathbb{R}^n)$. The trace operator

$$\gamma : \mathrm{BD}(U) \to L^1(\partial U;\mathbb{R}^m)$$

is continuous with respect to the norm topology but not with respect to the weak* topology of $\mathrm{BD}(U)$.

2.2 A few inequalities

We recall here the classical Korn and Poincaré inequalities and we prove a few extensions that will be useful in the rest of the work.

Theorem 2.2.1 (Korn inequality)**.** *Let $U \subset \mathbb{R}^N$ be an open, bounded, and connected set with Lipschitz boundary and let $1 < p < +\infty$. Then there exists*

$C > 0$ *such that*

$$\|\nabla u\|_{L^p} \le C(\|Eu\|_{L^p} + \|u\|_{L^p}) \quad \forall\, u \in W^{1,p}(U;\mathbb{R}^N)\,, \tag{2.3}$$

where Eu *is the symmetric gradient of* u*. Moreover, there exists* $C' > 0$ *such that for every* $u \in W^{1,p}(U;\mathbb{R}^N)$ *there is* $A \in M^{N\times N}_{skew}$ *such that*

$$\|\nabla u - A\|_{L^p} \le C'\|Eu\|_{L^p}\,. \tag{2.4}$$

For a proof of this classical result we refer to Theorem 3 and 8 in [12].

Lemma 2.2.1. *Let* $A = (a_{ij}) \in M^{N\times N}_{skew}$*. Then* $\operatorname{Rank} A \neq 1$.

Proof. Suppose by contradiction that $\operatorname{Rank} A = 1$. Let $A_1, \dots, A_n$ be the rows of A and let $i, j \in \{1, \dots, N\}$ with $i \neq j$. By assumption there exists $\lambda \in \mathbb{R}$ such that $A_i = \lambda A_j$. Hence, $a_{ij} = \lambda a_{jj} = 0$. Since A is skew-symmetric we deduce $a_{ji} = 0$. Then $A = 0$, a contradiction. ■

Theorem 2.2.2. *Let* $U \subset \mathbb{R}^N$ *be an open, bounded, and connected set with Lipschitz boundary and let* $1 < p < +\infty$*. Let* $\Gamma \subset \partial U$ *be an open subset (in the relative topology of* ∂U*) such that* $\mathscr{H}^{N-1}(\Gamma) > 0$*. Then there exists* $K > 0$ *such that*

$$\|\nabla u\|_{L^p} \le K(\|Eu\|_{L^p} + \|\gamma_\Gamma(u)\|_{L^p}) \quad \forall\, u \in W^{1,p}(U;\mathbb{R}^N)\,, \tag{2.5}$$

where $\gamma_\Gamma(u)$ *is the trace of* u *restricted to* Γ.

Proof. Suppose by contradiction that the thesis is false. Then for every $n \in \mathbb{N}$ there exists a function $u_n \in W^{1,p}(U;\mathbb{R}^N)$ such that

$$\|\nabla u_n\|_{L^p} > n(\|Eu_n\|_{L^p} + \|\gamma_\Gamma(u_n)\|_{L^p})\,. \tag{2.6}$$

Let $w_n := \dfrac{1}{\|u_n\|_{W^{1,p}}} u_n$, so that for every $n \in \mathbb{N}$ we have $\|w_n\|_{W^{1,p}} = 1$. Clearly, (2.6) holds with w_n in place of u_n. Up to a subsequence, there exists $w \in W^{1,p}(U;\mathbb{R}^N)$ such that

$$w_n \rightharpoonup w \quad \text{in } W^{1,p}(U;\mathbb{R}^N)\,.$$

By the compact embedding $W^{1,p}(U;\mathbb{R}^N) \subset\subset L^p(U;\mathbb{R}^N)$ we deduce that

$$w_n \to w \quad \text{in } L^p(U;\mathbb{R}^N)\,.$$

Similarly, by the compactness of the trace operator we infer

$$\gamma_\Gamma(w_n) \to \gamma_\Gamma(w) \quad \text{in } L^p(\partial U;\mathbb{R}^N)\,.$$

By (2.6) we have

$$\|Ew_n\|_{L^p} + \|\gamma_\Gamma(w_n)\|_{L^p} < \frac{1}{n}\|\nabla w_n\|_{L^p} \le \frac{1}{n} \quad \forall\, n \in \mathbb{N}\,,$$

hence, $\gamma_\Gamma(w) = 0$ and $Ew = 0$. By Theorem 2.2.1 there exists a matrix $A \in M^{3\times 3}_{skew}$ such that $\nabla w - A = 0$. Since U is connected, it must be $w(x) = Ax + b$ for a.e $x \in U$, where $b \in \mathbb{R}^N$ is a fixed vector. In particular, we have proved that Γ is a subset of the space of solutions of the linear system $Ax = -b$. By the Rouché-Capelli Theorem and Lemma 2.2.1 we know that the dimension of this space is not equal to $N-1$. Therefore, since $\mathscr{H}^{N-1}(\Gamma) > 0$, it must be $A = 0$ and $b = 0$, that is, $w = 0$. By Theorem 2.2.1 we obtain

$$\|\nabla w_n\|_{L^p} \leq C(\|Ew_n\|_{L^p} + \|w_n\|_{L^p}) \to 0\,,$$

which gives $\nabla w_n \to \nabla w = 0$ in $L^p(U;\mathbb{R}^N)$. This is a contradiction, since $w_n \to w$ in $W^{1,p}(U;\mathbb{R}^N)$, but $\|w_n\|_{W^{1,p}} \not\to \|w\|_{W^{1,p}}$. ∎

Theorem 2.2.3 (Poincaré inequality)**.** *Let $U \subset \mathbb{R}^N$ be an open, bounded, and connected set with Lipschitz boundary and let $1 \leq p < +\infty$. Then there exists $C_P > 0$ such that*

$$\|u\|_{L^p} \leq C_P\|\nabla u\|_{L^p} \quad \forall\, u \in W^{1,p}_0(U;\mathbb{R}^N)\,. \tag{2.7}$$

Theorem 2.2.4. *Let $U \subset \mathbb{R}^N$ be an open, bounded, and connected set with Lipschitz boundary and let $1 \leq p < +\infty$. Let $\Gamma \subset \partial U$ be an open subset (in the relative topology of ∂U) such that $\mathscr{H}^{N-1}(\Gamma) > 0$. Then there exists $C_P > 0$ such that*

$$\|u\|_{L^p} \leq C_P(\|\nabla u\|_{L^p} + \|\gamma_\Gamma(u)\|_{L^p}) \quad \forall\, u \in W^{1,p}(U;\mathbb{R}^N)\,. \tag{2.8}$$

Proof. Suppose by contradiction that the thesis is false. Then there is a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(U,\mathbb{R}^N)$ such that

$$\|u_n\|_{L^p} > n(\|\nabla u_n\|_{L^p} + \|\gamma_\Gamma(u_n)\|_{L^p}) \quad \forall\, n \in \mathbb{N}\,. \tag{2.9}$$

Up to normalization, we can suppose that $\|u_n\|_{W^{1,p}} = 1$ for every $n \in \mathbb{N}$. By compactness, up to a subsequence, there exists $u \in L^p(U;\mathbb{R}^N)$ such that $u_n \to u$ in $L^p(U;\mathbb{R}^N)$. By (2.9) we have

$$\|\nabla u_n\|_{L^p} + \|\gamma_\Gamma(u_n)\|_{L^p} < \frac{1}{n}\|u_n\|_{L^p} \leq \frac{1}{n} \quad \forall\, n \in \mathbb{N}\,,$$

hence $\nabla u_n \to 0$ in $L^p(U;\mathbb{R}^N)$ and $\gamma_\Gamma(u_n) \to 0$ in $L^p(\Gamma;\mathbb{R}^N)$. Therefore, we have proved that $u_n \to u$ in $W^{1,p}(U;\mathbb{R}^N)$ and by the continuity of the trace operator $\gamma(u) = 0$. In particular, $\nabla u = 0$, thus $u = b$ a.e in U with $b \in \mathbb{R}^N$ a fixed vector. By $\gamma_\Gamma(u) = 0$ we deduce $b = 0$ and this gives the contradiction. Indeed, we have $u_n \to 0$ in $W^{1,p}(U;\mathbb{R}^N)$, but $\|u_n\|_{W^{1,p}} \to 1$. ∎

We will also need the following inequality for functions of bounded deformation.

Theorem 2.2.5. *Let $U \subset \mathbb{R}^N$ be an open, bounded, and connected set with Lipschitz boundary. Let $\Gamma \subset \partial U$ be an open subset (in the relative topology of*

∂U) such that $\mathscr{H}^{N-1}(\Gamma) > 0$. Then there exists $C_{\mathrm{BD}} > 0$ such that

$$\|u\|_{\mathrm{BD}} \leq C_{\mathrm{BD}}(\|Eu\|_{\mathscr{M}_b} + \|\gamma_\Gamma(u)\|_{L^1}) \quad \forall\, u \in \mathrm{BD}(U)\,. \tag{2.10}$$

Proof. Owing to Proposition 2.3, item (ii), in [29] it is sufficient to prove that $\|\gamma_\Gamma(\cdot)\|_{L^1}$ is a continuous seminorm on $\mathrm{BD}(U)$ and a norm on the set of rigid motions given by

$$\mathscr{R} = \{u : U \to \mathbb{R}^N \ : \ u(x) = Ax + b,\ A \in M^{N\times N}_{skew},\ b \in \mathbb{R}^N\}\,.$$

By the continuity of the trace operator with respect to the strong topology of BD and its linearity the map $\|\gamma_\Gamma(\cdot)\|_{L^1}$ is a seminorm on $\mathrm{BD}(U)$ and thus, on $\mathscr{R}$. Let $u = Ax + b \in \mathscr{R}$ be such that $\|\gamma_\Gamma(u)\|_{L^1} = 0$, that is, $u = 0$ on Γ. Then, Γ is a subset of the space of solutions of the linear system $Ax = -b$. By the Rouché-Capelli Theorem and Lemma 2.2.1 we know that the dimension of this space is not equal to $N - 1$. Therefore, it must be $A = 0,\ b = 0$ concluding the proof. ∎

Finally, we state a recent generalization of the classical Korn inequality proved by Lewintan and Neff in [14].

Proposition 2.2.1. *Let $U \subset \mathbb{R}^3$ be an open, bounded, and connected set with Lipschitz boundary. Let $1 < p < \infty$ and let $f \in \mathscr{D}'(U; M^{3\times 3})$ be a distribution. If $f_{sym} \in L^p(U; M^{3\times3}_{sym})$ and $\mathrm{curl}(f) \in W^{-1,p}(U; M^{3\times3})$, then $f \in L^p(U; M^{3\times3})$. Moreover, there exist a constant $C > 0$, that depends only on p and U, such that*

$$\|f\|_{L^p} \leq C(\|f_{skew}\|_{W^{-1,p}} + \|f_{sym}\|_{L^p} + \|\,\mathrm{curl}(f)\|_{W^{-1,p}})\,.$$

Chapter 3

The model

3.1 Description of the model

In this section we give a brief description of the model proposed by Gurtin in [10]. Let $\Omega \subset \mathbb{R}^3$ be the reference configuration of the body and let $u : \Omega \to \mathbb{R}^3$ be the displacement (so that a point $x \in \Omega$ is mapped into the point $x + u(x)$ in the deformed configuration). As it is common in small-strain models, we assume the displacement gradient to be additively decomposed into an elastic and a plastic part, respectively:

$$\nabla u = H^e + H^p \,. \tag{3.1}$$

Since plastic deformations in metals are volume preserving, we assume H^p to be deviatoric, that is $\operatorname{Tr} H^p = 0$. In contrast with other models, such as the one proposed by Gurtin and Anand in [11] and studied in [6], here we do not impose that $H^p_{skew} = 0$, meaning that we admit plastic spin. In particular we do not restrict our attention only to the symmetric part of the displacement gradient. This is motivated by the macroscopic representation of the so-called Burgers vector, namely

$$B = \operatorname{curl}(H^p) = \operatorname{curl}(H^p_{sym}) + \operatorname{curl}(H^p_{skew}) \,. \tag{3.2}$$

The Burgers vector describes the macroscopic density of dislocations, which are defects in the crystalline structure at the microscopic level and are considered the main mechanism for plastic deformation. Clearly, H^p_{skew} is involved in the definition of B and needs to be considered in a theory that takes into account the Burgers vector.

Gurtin grounds his model on the principle of virtual power and introduces two microscopic stress tensors, called T^p and K, that perform work locally together with the temporal change of H^p and ∇H^p, respectively. Moreover, he defines an elastic stress tensor, named T, conjugate to the rate of change of H^e. Given a subbody $P \subset \Omega$, the power expended internally has the form

$$\mathcal{W}_{int}(P) = \int_P (T : \dot{H}^e + T^p : \dot{H}^p + K : \nabla \dot{H}^p)\, dx \,. \tag{3.3}$$

$\mathcal{W}_{int}(P)$ is balanced by the power expended on P by exterior forces. This is the result of a macroscopic body force f and a surface traction $t(\vec{n})$ with the addition of a microtraction $S(\vec{n})$ associated with $\dot{H}^p$ and can be written as

$$\mathcal{W}_{ext}(P) = \int_{\partial P} (t(\vec{n}) \cdot \dot{u} + S(\vec{n}) : \dot{H}^p) \; dA + \int_P f \cdot \dot{u} \; dx \,. \tag{3.4}$$

Since $\operatorname{Tr} H^p = 0$, we can suppose without loss of generality that T^p and $S(\vec{n})$ are deviatoric and, for the same reason, that K is deviatoric in the first two subscripts. By the power balance

$$\mathcal{W}_{int}(P) = \mathcal{W}_{ext}(P) \quad \forall \; P \subset \Omega \,, \tag{3.5}$$

we deduce the following conditions on the macroscopic traction and external forces:

$$t(\vec{n}) = T\vec{n} \,, \tag{3.6}$$

$$-\operatorname{div}(T) = f \,. \tag{3.7}$$

In particular, if a superficial force g acts on a portion of the boundary $\Gamma_N \subset \partial\Omega$ we can write

$$T\vec{n} = g \quad \text{on } \Gamma_N \,. \tag{3.8}$$

Similarly, one obtains the balance condition for the microscopic forces and traction, namely

$$T_D = T^p - \operatorname{div} K \,, \tag{3.9}$$

$$S(\vec{n}) = K\vec{n} \,, \tag{3.10}$$

where T_D denotes the deviatoric part of T. In this work we will assume null power expenditure on the boundary by K, that is,

$$S(\vec{n}) = K\vec{n} = 0 \quad \text{on } \partial\Omega \,. \tag{3.11}$$

Lastly, by frame-indifference it is possible to derive a well-known condition on the stress T, precisely its symmetry $T = T^T$.

The Burgers vector enters the model through the free energy ψ. Here we consider a quadratic free energy given by

$$\psi(H^e, B) := \mu |H^e_{sym,D}|^2 + \frac{1}{2} k |\operatorname{Tr} H^e_{sym}|^2 + \frac{\mu L^2}{2} |B|^2 \,, \tag{3.12}$$

where $\mu, k, L > 0$ are positive constants. In this context L has to be interpreted as a material length scale associated with the Burgers tensor. For any given subbody $P \subset \Omega$ the energy takes the integral form

$$\Psi(P) = \int_P \psi(H^e, B) \; dx \,. \tag{3.13}$$

The evolution of the triplet $(u(t), H^e(t), H^p(t))$ needs to satisfy a local thermodynamical requirement, namely the inequality

$$\dot{\psi} - T : \dot{H}^e_{sym} - T^p : \dot{H}^p - K : \nabla \dot{H}^p \leq 0\,, \tag{3.14}$$

where we used the symmetry of T. In order to satisfy inequality (3.14), Gurtin proposes the following constitutive equation for the stress tensors T:

$$T := \frac{\partial \psi}{\partial H^e_{sym}}\,, \tag{3.15}$$

that in our case rewrites as

$$T = \mathbb{C} H^e_{sym}\,, \tag{3.16}$$

where $\mathbb{C}$ is the tensor given by

$$\mathbb{C}A = 2\mu A_{sym,D} + k\,\mathrm{Tr}(A_{sym})\,\mathrm{Id} \quad \forall A \in M^{3\times 3}\,. \tag{3.17}$$

Then, letting

$$R := \frac{\partial \psi(H^e_{sym}, B)}{\partial B}\,, \tag{3.18}$$

Gurtin defines

$$(K_{diss})_{ijk} = K_{ijk} - \sum_{h=1}^{3} \epsilon_{hkj} R_{hi} + \frac{1}{3}\delta_{ij} \sum_{h,l=1}^{3} \epsilon_{hkl} R_{hl}\,, \tag{3.19}$$

where ϵ is the Levi-Civita symbol and δ is the usual Kronecker symbol. Clearly K_{diss} is deviatoric in the first two indices and, considering (3.15), inequality (3.14) takes the form

$$T^p : \dot{H}^p + K_{diss} : \nabla \dot{H}^p \geq 0\,. \tag{3.20}$$

To satisfy this condition Gurtin proposes the following constitutive equations for the stresses T^p and K^p_{diss} in the rate-independent case:

$$T^p := Y_0 \frac{\dot{H}^p_{sym} + \chi \dot{H}^p_{skew}}{d^p}\,, \tag{3.21}$$

$$K_{diss} := Y_0 \frac{h \nabla \dot{H}^p_{sym}}{d^p}\,, \tag{3.22}$$

where

$$d^p = \sqrt{|H^p_{sym}|^2 + \chi |H^p_{skew}|^2 + h^2 |\nabla H^p_{sym}|^2} \tag{3.23}$$

is the effective distortion-rate, $h, \chi > 0$ are positive constants, and $Y_0 > 0$ is the yield strength. Notice that with these choices we can suppose that K_{diss} is symmetric in the first two subscripts. As a remark we can see that when $\chi, h \to 0$ the framework reduces to the one of the Von Mises flow rule in perfect plasticity.

In this scenario the stress tensors have to satisfy the constraint

$$\sqrt{|T^p_{sym}|^2 + \frac{1}{\chi}|T^p_{skew}|^2 + \frac{1}{h^2}|K_{diss}|^2} \le Y_0\,, \tag{3.24}$$

where the equality can be interpreted as the stresses lying on the yield surface. For this very reason equations (3.21) and (3.22) will be valid when equality holds in (3.24). Viceversa, if the stresses lie well inside the set bounded by the yield surface, no changes occur in the plastic strain, that is, $\dot{H}^p = 0,\ \nabla\dot{H}^p_{sym} = 0$. With these choices the thermodynamical requirement (3.14) is satisfied.

3.2 The functional setting

3.2.1 Reference configuration and boundary conditions

We now introduce the precise mathematical setting of the problem. We assume the reference configuration of the body to be an open, bounded, and connected set with Lipschitz boundary. The Dirichlet part of the boundary is assumed to be a non-empty open subset $\Gamma_D \subset \partial\Omega$ (here open refers to the relative topology in $\partial\Omega$). The set $\Gamma_N := \partial\Omega\backslash\overline{\Gamma}_D$ is the Neumann part of the boundary. In particular, $\mathscr{H}^2(\Gamma_D) > 0$, where $\mathscr{H}^2$ is the 2-dimensional Hausdorff measure.

The forces

$$f \in \mathrm{AC}(0,T;L^3(\Omega;\mathbb{R}^3))\,,\ g \in \mathrm{AC}(0,T;L^3(\Gamma_N;\mathbb{R}^3)) \tag{3.25}$$

will be prescribed on Ω and Γ_N, respectively. On the Dirichlet part of the boundary a displacement

$$w \in \mathrm{AC}(0,T;H^1(\Omega;\mathbb{R}^3)) \tag{3.26}$$

is imposed. As a remark note that w is defined on the whole set Ω, but the boundary condition is prescribed only on Γ_D via the trace operator.

For every time $t \in [0,T]$ we define the operator

$$\mathcal{L}(t) : W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3) \to \mathbb{R} : u \mapsto \int_\Omega f(t)\cdot u\ dx + \int_{\Gamma_N} g(t)\cdot u\ d\mathscr{H}^2\,. \tag{3.27}$$

Clearly, $\mathcal{L}(t)$ is linear and continuous, hence we have $\mathcal{L}(t) \in W^{-1,3}(\Omega;\mathbb{R}^3)$. In addition, by the absolute continuity of f and g it follows that the operator $\mathcal{L}$ is absolutely continuous with values in $W^{-1,3}(\Omega;\mathbb{R}^3)$. We will write $\langle\mathcal{L}(t), u\rangle$ when $\mathcal{L}(t)$ is applied to u.

Since f and g are absolutely continuous with values in a reflexive space, the time derivatives $\dot{f}$ and $\dot{g}$ are well defined almost everywhere. For almost every $t \in [0,T]$ we will denote by $\dot{\mathcal{L}}(t)$ the continuous and linear operator given by

$$\dot{\mathcal{L}}(t) : W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3) \to \mathbb{R} : u \mapsto \int_\Omega \dot{f}(t)\cdot u\ dx + \int_{\Gamma_N} \dot{g}(t)\cdot u\ d\mathscr{H}^2\,. \tag{3.28}$$

3.2.2 Admissible configurations

We will keep the notation already used and we will denote by u the displacement and by H^e, H^p the elastic and plastic part of the strain, respectively.

Definition 3.2.1. *Let $z \in H^1(\Omega;\mathbb{R}^3)$. A triplet (u, H^e, H^p) is said to be admissible for the boundary value z if and only if:*

$$\begin{aligned} &u \in W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3)\,, && H^e_{sym} \in L^2(\Omega; M^{3\times3}_{sym})\,, \\ &H^p_{sym} \in \mathrm{BV}(\Omega; M^{3\times3}_{D,sym})\,, && H^p_{skew} \in \mathscr{M}_b(\Omega; M^{3\times3}_{skew})\,, \\ &\qquad \operatorname{curl} H^p \in L^2(\Omega; M^{3\times3})\,, \end{aligned} \tag{3.29}$$

$$\nabla u = H^e + H^p\,, \tag{3.30}$$

$$u = z \text{ in } L^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^3)\,, \tag{3.31}$$

where equation (3.31) has to be intended in the sense of trace. We will denote by $\mathcal{A}(z)$ the set of admissible triplets (u, H^e, H^p) for the boundary value z.

From now on, when we will speak of an admissible plastic strain H^p we will imply that there is a boundary value $z \in H^1(\Omega;\mathbb{R}^3)$ and an admissible triplet $(v, e, p) \in \mathcal{A}(z)$ such that $p = H^p$. Similarly, we will speak of an admissible time-dependent plastic strain whenever we will have a map $t \mapsto H^p(t)$ such that $H^p(t)$ is an admissible plastic strain for every $t \in [0, T]$.

Owing to Proposition 2.2.1, admissible triplets have more regularity than the one prescribed by their definition. More precisely, we have the following.

Proposition 3.2.1. *Let $z \in H^1(\Omega;\mathbb{R}^3)$ and let $(u, H^e, H^p) \in \mathcal{A}(z)$. Then*

$$H^e\,, H^p \in L^{\frac{3}{2}}(\Omega; M^{3\times3})\,. \tag{3.32}$$

Moreover, there exists $C > 0$, which depends only on Ω, such that

$$\|H^p\|_{L^{\frac{3}{2}}} \le C(\|H^p_{skew}\|_{\mathscr{M}_b} + \|H^p_{sym}\|_{\mathrm{BV}} + \|\operatorname{curl} H^p\|_{L^2})\,. \tag{3.33}$$

In particular, from this result we infer that

$$H^p_{skew} = H^p - H^p_{sym} \in L^{\frac{3}{2}}(\Omega; M^{3\times3})\,. \tag{3.34}$$

Hence, H^p_{skew} is absolutely continuous with respect to the Lebesgue measure.

3.2.3 The dissipation functional

To enforce the constraint (3.24) we introduce the functional, defined for sufficiently regular plastic strains p

$$p \mapsto Y_0 \int_\Omega \sqrt{|p_{sym}|^2 + \chi|p_{skew}|^2 + h^2|\nabla p_{sym}|^2}\; dx\,. \tag{3.35}$$

Following [15] and [6] we relax the functional (3.35) using the notion of convex functions of measure introduced in [9]. First, we introduce the function

$$\mathcal{F} : M^{3\times3}_{sym} \times M^{3\times3}_{skew} \times M^{3\times3\times3}_{sym} \to \mathbb{R} : \quad (x, y, z) \mapsto \sqrt{|x|^2 + \chi|y|^2 + h^2|z|^2}\,.$$

For $p \in \mathscr{M}_b(\Omega; M^{3\times3}_D)$ with $p_{sym} \in BV(\Omega; M^{3\times3}_{D,sym})$ let λ be the measure on Ω defined by

$$\lambda = (p_{sym}, p_{skew}, \nabla p_{sym})\,. \tag{3.36}$$

We define

$$\mathcal{H} : p \mapsto Y_0 \int_\Omega \mathcal{F}\left(\frac{\lambda}{|\lambda|}\right) d|\lambda|\,, \tag{3.37}$$

where $|\lambda|$ is the total variation measure of λ and $\dfrac{\lambda}{|\lambda|}$ is the Radon-Nikodým derivative of λ with respect to $|\lambda|$.

Let H^p be an admissible plastic strain and let λ be the measure defined as in (3.36) with $p = H^p$. We can decompose λ in its absolutely continuous and singular parts with respect to the Lebesgue measure. By the regularity of H^p it is clear that

$$\lambda^a = (H^p_{sym}, H^p_{skew}, \nabla^a H^p_{sym}) \quad \text{and} \quad \lambda^s = (0, 0, \nabla^s H^p_{sym})\,. \tag{3.38}$$

Hence, by Theorem 2 in [9], it follows that

$$\begin{aligned} \mathcal{H}(H^p) =& Y_0 \int_\Omega \sqrt{|H^p_{sym}|^2 + \chi|H^p_{skew}|^2 + h^2|\nabla^a H^p_{sym}|^2}\, dx \\ &+ hY_0|\nabla^s H^p_{sym}|(\Omega)\,, \end{aligned} \tag{3.39}$$

where $|\nabla^s H^p_{sym}|(\Omega)$ is the total variation over Ω of the singular part of the measure ∇H^p_{sym}. Sometimes we will shorten the decomposition (3.39) and write

$$\mathcal{H}(H^p) = \mathcal{H}_1(H^p) + \mathcal{H}_2(H^p)\,. \tag{3.40}$$

We will often use the following lower semicontinuity result.

Proposition 3.2.2. *Let $(H^p_n)_{n\in\mathbb{N}}$ be a sequence of admissible plastic strains such that*

$$\begin{aligned} H^p_{n,sym} &\overset{*}{\rightharpoonup} H^p_{sym} \quad && \text{in } \mathrm{BV}(\Omega; M^{3\times3}_{sym})\,, \\ H^p_{n,skew} &\overset{*}{\rightharpoonup} H^p_{skew} && \text{in } \mathscr{M}_b(\Omega; M^{3\times3}_{skew})\,. \end{aligned}$$

Then

$$\mathcal{H}(H^p) \le \liminf_{n\to\infty} \mathcal{H}(H^p_n)\,.$$

Proof. Thanks to Theorem 3 in [9] it is sufficient to prove that the measure λ_n, defined as in (3.36) with $p = H^p_n$, converges weakly* in the sense of measures to $\lambda = (H^p_{sym}, H^p_{skew}, \nabla H^p_{sym})$. This assertion follows immediately from the hypotheses. ∎

Another important property of $\mathcal{H}$ is its continuity along strongly converging sequences, as proved in the following proposition.

Proposition 3.2.3. *Let* $(H^p_n)_{n\in\mathbb{N}}$ *be a sequence of admissible plastic strains such that*

$$\begin{aligned} H^p_{n,sym} &\to H^p_{sym} && \text{in } \mathrm{BV}(\Omega; M^{3\times3}_{sym}), \\ H^p_{n,skew} &\to H^p_{skew} && \text{in } \mathscr{M}_b(\Omega; M^{3\times3}_{skew}). \end{aligned}$$

Then

$$\lim_{n\to\infty} \mathcal{H}(H^p_n - H^p) = 0.$$

Proof. By the regularity of the admissible plastic strains and the supposed convergences we have that

$$\begin{aligned} H^p_{n,sym} &\to H^p_{sym} && \text{in } L^1(\Omega; M^{3\times3}_{sym}), \\ \nabla^a H^p_{n,sym} &\to \nabla^a H^p_{sym} && \text{in } L^1(\Omega; M^{3\times3}_{sym}), \\ \nabla^s H^p_{n,sym} &\to \nabla^s H^p_{sym} && \text{in } \mathscr{M}_b(\Omega; M^{3\times3}_{sym}), \\ H^p_{n,skew} &\to H^p_{skew} && \text{in } L^1(\Omega; M^{3\times3}_{skew}). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{H}(H^p_n - H^p) &\le Y_0 \|H^p_{n,sym} - H^p_{sym}\|_{L^1} + Y_0\sqrt{\chi}\|H^p_{n,skew} - H^p_{skew}\|_{L^1} \\ &+ hY_0\|\nabla^a H^p_{n,sym} - \nabla^a H^p_{sym}\|_{L^1} + hY_0|\nabla^s H^p_{n,sym} - \nabla^s H^p_{sym}|(\Omega) \to 0. \end{aligned}$$

■

Given an admissible time-dependent plastic strain $t \mapsto H^p(t)$ we will use throughout the paper the following notation for the $\mathcal{H}$-variation:

$$\mathcal{V}_{\mathcal{H}}(H^p; 0, t) = \sup\left\{ \sum_{i=1}^n \mathcal{H}(H^p(t_i) - H^p(t_{i-1})) \ : \ n \in \mathbb{N} \quad \text{and} \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t \right\}. \tag{3.41}$$

$\mathcal{V}_{\mathcal{H}}$ satisfies the following lower semicontinuity result.

Proposition 3.2.4. *Let* $(H^p_n)_{n\in\mathbb{N}}$ *be a sequence of admissible time-dependent plastic strains such that for every* $t \in [0,T]$ *the following convergences hold:*

$$\begin{aligned} H^p_{n,sym}(t) &\overset{*}{\rightharpoonup} H^p_{sym}(t) && \text{in } \mathrm{BV}(\Omega; M^{3\times3}_{sym}), \\ H^p_{n,skew}(t) &\overset{*}{\rightharpoonup} H^p_{skew}(t) && \text{in } \mathscr{M}_b(\Omega; M^{3\times3}_{skew}). \end{aligned}$$

Then

$$\mathcal{V}_{\mathcal{H}}(H^p; 0, t) \le \liminf_{n\to\infty} \mathcal{V}_{\mathcal{H}}(H^p_n; 0, t) \quad \forall\, t \in [0,T].$$

Proof. Let $n \in \mathbb{N}$ and $t \in [0,T]$. Let $\{t_0, \ldots, t_k\}$ be a partition of the interval

$[0,t]$. By definition of $\mathcal{H}$-variation and by Proposition 3.2.2 we have

$$\liminf_{n\to\infty} \mathcal{V}_{\mathcal{H}}(H^p_n;0,t) \geq \liminf_{n\to\infty} \sum_{i=1}^{k} \mathcal{H}(H^p_n(t_i) - H^p_n(t_{i-1})) \geq \sum_{i=1}^{k} \mathcal{H}(H^p(t_i) - H^p(t_{i-1})) \,. \tag{3.42}$$

Passing to the supremum over all partitions of the interval $[0,t]$ gives the thesis. ■

3.2.4 The free energy

We will consider the quadratic free energy defined in (3.12). Given an admissible triplet (u, H^e, H^p) we will write

$$\Psi(H^e_{sym}, \mathrm{curl}(H^p)) = \frac{1}{2}\int_\Omega \mathbb{C}H^e_{sym} : H^e_{sym}\; dx + \frac{\mu L^2}{2}\int_\Omega |\,\mathrm{curl}(H^p)|^2\; dx\,, \tag{3.43}$$

where $\mathbb{C}$ is the symmetric tensor in (3.17). Since $k > 0$ there are $\alpha_{\mathbb{C}}, \beta_{\mathbb{C}} > 0$ such that

$$\alpha_{\mathbb{C}}|M|^2 \leq \mathbb{C}M : M \leq \beta_{\mathbb{C}}|M|^2 \quad \forall\, M \in M^{3\times3}_{sym}\,. \tag{3.44}$$

By (3.44) it follows that

$$|\mathbb{C}M| \leq \beta_{\mathbb{C}}|M| \quad \forall\, M \in M^{3\times3}_{sym}\,. \tag{3.45}$$

We will often use the notation

$$\Psi(H^e, \mathrm{curl}(H^p)) = \Psi_1(H^e_{sym}) + \Psi_2(\mathrm{curl}(H^p)) \tag{3.46}$$

to shorten the expression (3.43).

3.2.5 Safe-load condition

As in the case of perfect plasticity (see [15]) we need to assume a so-called safe-load condition on the applied forces. More precisely, for the rest of the work we will assume the existence of a function

$$\rho \in \mathrm{AC}(0,T; L^3_{\mathrm{div}}(\Omega; M^{3\times3}_{sym})) \text{ with } \rho_D \in \mathrm{AC}(0,T; L^\infty(\Omega; M^{3\times3}_{D,sym})) \tag{3.47}$$

such that, for every $t \in [0,T]$

$$\begin{cases} -\operatorname{div}\rho(t) = f(t) & \text{in } L^3(\Omega;\mathbb{R}^3)\,, \\ \gamma_\nu(\rho(t)) = g(t) & \text{on } \Gamma_N\,, \end{cases} \tag{3.48}$$

where γ_ν is the normal trace. The second equation in (3.48) has to be intended in the following sense: for every $\psi \in W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega, \mathbb{R}^3)$ such that $\psi = 0$ $\mathscr{H}^2$-a.e on

Γ_D

$$\langle \gamma_\nu(\rho(t)), \psi \rangle = \int_{\Gamma_N} g \cdot \psi \; d\mathscr{H}^2 \,, \tag{3.49}$$

where the duality at the left-hand side is the one between $W^{-\frac{1}{3},3}(\partial\Omega;\mathbb{R}^3)$ and $W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega;\mathbb{R}^3)$.

Moreover, we will suppose the existence of a constant $M > 0$ such that for every $t \in [0,T]$

$$|A + \rho_D(t)| \leq Y_0 \quad \text{a.e in } \Omega \quad \forall\, A \in M_D^{3\times 3} \text{ with } |A| \leq M\,. \tag{3.50}$$

We will assume that $M < Y_0$ without loss of generality. The main consequence of the safe-load condition is the following proposition.

Proposition 3.2.5. *Let $z \in H^1(\Omega;\mathbb{R}^3)$. Suppose that a function ρ exists satisfying (3.47), (3.48), and (3.50). Then, for every $t \in [0,T]$ and for every triplet $(u, H^e, H^p) \in \mathcal{A}(z)$ we have*

$$\begin{aligned} \langle \mathcal{L}(t), u \rangle =& \langle \mathcal{L}(t), z \rangle - \int_\Omega \rho(t) : Ez \; dx \\ &+ \int_\Omega \rho(t) : H^e_{sym} \; dx + \int_\Omega \rho_D(t) : H^p_{sym} \; dx\,. \end{aligned} \tag{3.51}$$

Proof. Let $t \in [0,T]$. By definition of the operator $\mathcal{L}(t)$ we have

$$\langle \mathcal{L}(t), u - z \rangle = \int_\Omega f(t) \cdot (u - z) \; dx + \int_{\Gamma_N} g(t) \cdot (u - z) \; d\mathscr{H}^2\,. \tag{3.52}$$

Since $\rho(t)$ satisfies (3.48), integrating by parts yields

$$\begin{aligned} \langle \mathcal{L}(t), u - z \rangle = & \int_\Omega \rho(t) : \nabla(u - z) \; dx - \langle \gamma_\nu(\rho(t)), u - z \rangle \\ &+ \int_{\Gamma_N} g(t) \cdot (u - z) \; d\mathscr{H}^2\,. \end{aligned} \tag{3.53}$$

Note that $u - z = 0$ $\mathscr{H}^2$-a.e on Γ_D, hence by (3.49)

$$\int_{\Gamma_N} g(t) \cdot (u - z) \; d\mathscr{H}^2 - \langle \gamma_\nu(\rho(t)), u - z \rangle = 0\,.$$

Finally, by the admissibility of (u, H^e, H^p) and the symmetry of $\rho(t)$, equation (3.53) can be rewritten as follows:

$$\langle \mathcal{L}(t), u - z \rangle = \int_\Omega \rho(t) : H^e_{sym} \; dx + \int_\Omega \rho_D(t) : H^p_{sym} \; dx - \int_\Omega \rho(t) : Ez \; dx\,,$$

concluding the proof. ∎

In a similar fashion one can prove the same result for the operator $\dot{\mathcal{L}}$. As a remark note that assumptions (3.47) grants the existence of the time derivatives $\dot\rho$ and $\dot\rho_D$ almost everywhere.

Proposition 3.2.6. *Let $z \in H^1(\Omega;\mathbb{R}^3)$. Suppose that a function ρ exists satisfying (3.47), (3.48), and (3.50). Then, for almost every $t \in [0,T]$ and for every triplet $(u, H^e, H^p) \in \mathcal{A}(z)$ we have*

$$\begin{aligned}\langle \dot{\mathcal{L}}(t), u\rangle =& \langle \dot{\mathcal{L}}(t), z\rangle - \int_\Omega \dot\rho(t) : Ez\, dx \\ &+ \int_\Omega \dot\rho(t) : H^e_{sym}\, dx + \int_\Omega \dot\rho_D(t) : H^p_{sym}\, dx\,. \end{aligned} \tag{3.54}$$

3.2.6 Energetic solutions

We will prove existence of solutions for the Gurtin model using the energetic approach to rate-independent processes developed in [16, 17, 19, 20, 21]. We state here the definition of a solution to the Gurtin model in this framework.

Definition 3.2.2. *Let f, g be as in (3.25) and w be as in (3.26). A map*

$$\begin{gathered}\Phi : [0,T] \to W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3) \times L^{\frac{3}{2}}(\Omega; M^{3\times3}) \times L^{\frac{3}{2}}(\Omega; M^{3\times3}_D) : \\ t \mapsto (u(t), H^e(t), H^p(t))\end{gathered}$$

is an energetic solution to the Gurtin model if the following properties hold true:

- *Admissibility:*

$$(u(t), H^e(t), H^p(t)) \in \mathcal{A}(w(t)) \quad \forall\; t \in [0,T]\,, \tag{3.55}$$

- *Global stability:*

$$\begin{aligned}\mathcal{E}(t) \le \Psi_1(e_{sym}) + \Psi_2(\mathrm{curl}(p)) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(p - H^p(t)) \\ \forall\; (v,e,p) \in \mathcal{A}(w(t)) \quad \forall\; t \in [0,T]\,,\end{aligned} \tag{3.56}$$

- *Bounded variation: H^p_{sym} and H^p_{skew} have bounded variation as maps*

$$\begin{aligned} H^p_{sym} &: [0,T] \to \mathrm{BV}(\Omega; M^{3\times3}_{D,sym})\,, \\ H^p_{skew} &: [0,T] \to L^1(\Omega; M^{3\times3}_{skew})\,, \end{aligned}$$

- *Energy balance:*

$$\begin{aligned}\mathcal{E}(t) + \mathcal{V}_{\mathcal{H}}(H^p; 0, t) = \mathcal{E}(0) + \int_0^t \int_\Omega \mathbb{C}H^e_{sym}(\tau) : E\dot w(\tau)\, dx d\tau \\ - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot w(\tau)\rangle\, d\tau \quad \forall\; t \in [0,T]\,,\end{aligned} \tag{3.57}$$

where

$$\mathcal{E}(t) := \Psi_1(H^e_{sym}(t)) + \Psi_2(\mathrm{curl}(H^p(t))) - \langle \mathcal{L}(t), u(t)\rangle\,. \tag{3.58}$$

Remark 3.2.1. The assumption $H^e(t) \in L^{\frac{3}{2}}(\Omega; M^{3\times3})$ for every $t \in [0,T]$ is non-restrictive. Indeed, by the admissibility (3.55) and Proposition 3.2.1 it follows

$$H^e_{skew}(t) = \nabla u - H^e_{sym} - H^p \in L^{\frac{3}{2}}(\Omega; M^{3\times 3}_{skew})\,.$$

Moreover, H^p_{skew} has bounded variation with values in the spaces $L^1(\Omega; M^{3\times 3}_{skew})$ or $\mathscr{M}_b(\Omega; M^{3\times 3}_{skew})$, equivalently.

To make sense of the energy balance (3.57), measurability of u and H^e_{sym} as maps from $[0,T]$ into $L^{\frac{3}{2}}(\Omega;\mathbb{R}^3)$ and $L^2(\Omega; M^{3\times 3}_{sym})$, respectively, is needed. We now prove that admissibility, global stability, and the bounded variation of H^p_{sym} and H^p_{skew} are sufficient to guarantee it.

Lemma 3.2.1. *Let*

$$t \mapsto (u(t), H^e(t), H^p(t))$$

be a map such that admissibility (3.55) and global stability (3.56) hold. Then for every $t \in [0,T]$ and for every $(v,e,p) \in \mathcal{A}(0)$

$$\left| \int_\Omega \mathbb{C} H^e_{sym}(t) : e_{sym}\; dx + \mu L^2 \int_\Omega \operatorname{curl}(H^p(t)) : \operatorname{curl}(p)\; dx - \langle \mathcal{L}(t), v\rangle \right| \le \mathcal{H}(p)\,.$$

Proof. Let $\varepsilon \in \mathbb{R}$ and $(v,e,p) \in \mathcal{A}(0)$. Let $t \in [0,T]$. The triplet

$$(u(t) + \varepsilon v, H^e(t) + \varepsilon e, H^p(t) + \varepsilon p)$$

is admissible for the boundary value $w(t)$. By the global stability condition (3.56) we have

$$\begin{aligned} &\Psi_1(H^e_{sym}(t)) + \Psi_2(\operatorname{curl}(H^p(t))) - \langle \mathcal{L}(t), u(t)\rangle \le \Psi_1(H^e_{sym}(t) + \varepsilon e_{sym}) \\ &+ \Psi_2(\operatorname{curl}(H^p(t)) + \varepsilon \operatorname{curl}(p)) - \langle \mathcal{L}(t), u(t) + \varepsilon v\rangle + \mathcal{H}(\varepsilon p)\,. \end{aligned} \tag{3.59}$$

From (3.59) we deduce that

$$\begin{aligned} &\varepsilon^2 \Psi_1(e_{sym}) + \varepsilon^2 \Psi_2(\operatorname{curl}(p)) + \varepsilon \int_\Omega \mathbb{C} H^e_{sym}(t) : e_{sym}\; dx \\ &+ \varepsilon \mu L^2 \int_\Omega \operatorname{curl}(H^p(t)) : \operatorname{curl}(p)\; dx - \varepsilon \langle \mathcal{L}(t), v\rangle + \mathcal{H}(\varepsilon p) \ge 0\,. \end{aligned} \tag{3.60}$$

By the positive homogeneity of $\mathcal{H}$, dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0^+$ in (3.60), we obtain

$$\int_\Omega \mathbb{C} H^e_{sym}(t) : e_{sym}\; dx + \mu L^2 \int_\Omega \operatorname{curl}(H^p(t)) : \operatorname{curl}(p)\; dx - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(p) \ge 0\,.$$

If instead we consider $\varepsilon < 0$ and let $\varepsilon \to 0^-$, then

$$\int_\Omega \mathbb{C} H^e_{sym}(t) : e_{sym}\; dx + \mu L^2 \int_\Omega \operatorname{curl}(H^p(t)) : \operatorname{curl}(p)\; dx - \langle \mathcal{L}(t), v\rangle - \mathcal{H}(p) \le 0\,.$$

The thesis follows by combining the last two equations. ∎

Proposition 3.2.7. *Let*

$$t \mapsto (u(t), H^e(t), H^p(t))$$

be a map such that admissibility (3.55) and global stability (3.56) hold. Then there exists $C > 0$ such that for every $t_1, t_2 \in [0, T]$

$$\begin{aligned}
&\|H^e_{sym}(t_2) - H^e_{sym}(t_1)\|^2_{L^2} + \|\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1))\|^2_{L^2} \\
&\leq C \Big[\|H^p_{sym}(t_2) - H^p_{sym}(t_1)\|_{BV} + \|H^p_{skew}(t_2) - H^p_{skew}(t_1)\|_{L^1} \\
&+ \|\mathcal{L}(t_2) - \mathcal{L}(t_1)\|^2_{W^{-1,3}} + \|w(t_2) - w(t_1)\|^2_{H^1} \Big] . \qquad (3.61)
\end{aligned}$$

Proof. The triplet (v, e, p) defined by

$$\begin{aligned}
v &= u(t_2) - u(t_1) - (w(t_2) - w(t_1)) , \\
e &= H^e(t_2) - H^e(t_1) - (\nabla w(t_2) - \nabla w(t_1)) , \\
p &= H^p(t_2) - H^p(t_1) ,
\end{aligned}$$

is admissible for the zero boundary value. Therefore, by Lemma 3.2.1 we have

$$\begin{aligned}
&\int_\Omega \mathbb{C} H^e_{sym}(t) : (H^e_{sym}(t_2) - H^e_{sym}(t_1)) \, dx \\
&+ \mu L^2 \int_\Omega \operatorname{curl}(H^p(t)) : (\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1))) \, dx \\
&\leq \mathcal{H}(H^p(t_2) - H^p(t_1)) + \int_\Omega \mathbb{C} H^e_{sym}(t) : (Ew(t_2) - Ew(t_1)) \, dx \\
&+ \langle \mathcal{L}(t), u(t_2) - u(t_1) - (w(t_2) - w(t_1)) \rangle \quad \forall\, t \in [0, T] . \qquad (3.62)
\end{aligned}$$

Using the same argument with the triplet $(-v, -e, -p)$, we obtain

$$\begin{aligned}
&\int_\Omega \mathbb{C} H^e_{sym}(t) : (H^e_{sym}(t_1) - H^e_{sym}(t_2)) \, dx \\
&+ \mu L^2 \int_\Omega \operatorname{curl}(H^p(t)) : (\operatorname{curl}(H^p(t_1)) - \operatorname{curl}(H^p(t_2))) \, dx \\
&\leq \mathcal{H}(H^p(t_2) - H^p(t_1)) + \int_\Omega \mathbb{C} H^e_{sym}(t) : (Ew(t_1) - Ew(t_2)) \, dx \\
&+ \langle \mathcal{L}(t), u(t_1) - u(t_2) - (w(t_1) - w(t_2)) \rangle \quad \forall\, t \in [0, T] . \qquad (3.63)
\end{aligned}$$

Combining (3.62) with $t = t_2$ and (3.63) with $t = t_1$ we deduce that

$$\begin{aligned}
&2\Psi_1(H^e_{sym}(t_2) - H^e_{sym}(t_1)) + 2\Psi_2(\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1))) \\
&\leq 2\mathcal{H}(H^p(t_2) - H^p(t_1)) + \int_\Omega \mathbb{C}(H^e_{sym}(t_2) - H^e_{sym}(t_1)) : (Ew(t_2) - Ew(t_1)) \, dx \\
&+ \langle \mathcal{L}(t_2) - \mathcal{L}(t_1), u(t_2) - u(t_1) - (w(t_2) - w(t_1)) \rangle . \qquad (3.64)
\end{aligned}$$

Note that by definition of $\mathcal{H}$ there exists $C > 0$ such that

$$\begin{aligned}
2\mathcal{H}(H^p(t_2) - H^p(t_1)) &\leq C \|H^p_{sym}(t_2) - H^p_{sym}(t_1)\|_{BV} \\
&+ C \|H^p_{skew}(t_2) - H^p_{skew}(t_1)\|_{L^1} . \qquad (3.65)
\end{aligned}$$

By the Hölder inequality, the coercivity properties (3.44)-(3.45), (3.64), and

(3.65) we obtain

$$
\begin{aligned}
&\alpha_{\mathbb{C}}\|H^e_{sym}(t_2) - H^e_{sym}(t_1)\|^2_{L^2} + \mu L^2\|\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1))\|^2_{L^2}\\
&\leq C\|H^p_{sym}(t_2) - H^p_{sym}(t_1)\|_{BV} + C\|H^p_{skew}(t_2) - H^p_{skew}(t_1)\|_{\mathscr{M}_b}\\
&+ \beta_{\mathbb{C}}\|H^e_{sym}(t_2) - H^e_{sym}(t_1)\|_{L^2}\|w(t_2) - w(t_1)\|_{H^1}\\
&+ \|\mathcal{L}(t_2) - \mathcal{L}(t_1)\|_{W^{-1,3}}\|u(t_2) - u(t_1)\|_{W^{1,\frac{3}{2}}}\\
&+ \|\mathcal{L}(t_2) - \mathcal{L}(t_1)\|_{W^{-1,3}}\|w(t_2) - w(t_1)\|_{W^{1,\frac{3}{2}}}\,.
\end{aligned}
\tag{3.66}
$$

We now focus on the term involving u. By Theorem 2.2.2 and Theorem 2.2.4 we deduce that there exists $C' > 0$ such that

$$
\begin{aligned}
&\|u(t_2) - u(t_1)\|_{W^{1,\frac{3}{2}}} \leq C'\|H^e_{sym}(t_2) - H^e_{sym}(t_1)\|_{L^2}\\
&+ C'\|H^p_{sym}(t_2) - H^p_{sym}(t_1)\|_{BV} + C'\|w(t_2) - w(t_1)\|_{H^1}\,.
\end{aligned}
\tag{3.67}
$$

Note that, since $\mathcal{L}$ is absolutely continuous, we have

$$
\sup_{t\in[0,T]} \|\mathcal{L}(t)\|_{W^{-1,3}} < +\infty\,. \tag{3.68}
$$

Combining (3.66)-(3.68) we have shown that there exists a constant $C > 0$ such that

$$
\begin{aligned}
&\|H^e_{sym}(t_2) - H^e_{sym}(t_1)\|^2_{L^2} + \|\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1))\|^2_{L^2}\\
&\leq C\|H^p_{sym}(t_2) - H^p_{sym}(t_1)\|_{BV} + C\|H^p_{skew}(t_2) - H^p_{skew}(t_1)\|_{L^1}\\
&+ C\|H^e_{sym}(t_2) - H^e_{sym}(t_1)\|_{L^2}\|w(t_2) - w(t_1)\|_{H^1}\\
&+ C\|H^e_{sym}(t_2) - H^e_{sym}(t_1)\|_{L^2}\|\mathcal{L}(t_2) - \mathcal{L}(t_1)\|_{W^{-1,3}}\\
&+ C\|\mathcal{L}(t_2) - \mathcal{L}(t_1)\|_{W^{-1,3}}\|w(t_2) - w(t_1)\|_{H^1}\,.
\end{aligned}
\tag{3.69}
$$

Therefore, by the Cauchy inequality, up to changing the constant C, we deduce (3.61) ■

Suppose that a map

$$
t \mapsto (u(t), H^e(t), H^p(t))
$$

satisfies admissibility (3.55) and global stability (3.56). Moreover, assume that the maps

$$
\begin{aligned}
H^p_{sym} &: [0,T] \to \mathrm{BV}(\Omega; M^{3\times 3}_{D,sym})\,,\\
H^p_{skew} &: [0,T] \to L^1(\Omega; M^{3\times 3}_{skew})\,,
\end{aligned}
$$

have bounded variation. Then H^p_{sym} and H^p_{skew} are continuous almost everywhere. Let $t \in [0,T]$ be a point of continuity for H^p_{sym} and H^p_{skew}. By Proposition 3.2.7, t is a point of continuity for the map

$$
H^e_{sym} : [0,T] \to L^2(\Omega; M^{3\times 3}_{sym})\,.
$$

Therefore, by Theorem 2.2.2 and Theorem 2.2.4 t is also a point of continuity for

$$u : [0, T] \to W^{1,\frac{3}{2}}(\Omega; \mathbb{R}^3) .$$

Hence u and H^e_{sym} are measurable as functions on $[0, T]$ with values in the corresponding spaces.

The next chapter will be devoted to proving the existence of an energetic solution to the Gurtin model.

Chapter 4

Existence of an energetic solution

In this chapter we will show the existence of an energetic solution to the Gurtin model. We will construct piecewise-constant-in-time approximate solutions by discretizing the interval $[0, T]$ and by solving, at each discrete time, a minimization problem able to enforce the global stability. Then we will show the convergence of such approximated solutions to a proper energetic solution. The chapter is organized as follows: in the first section we discuss the solvability of the incremental minimization problems; in the second section we will construct the approximate solutions and investigate their properties; in the last section we will pass to the limit in the discretization parameter and exhibit the existence of an energetic solution to the Gurtin model.

4.1 The minimization problem

To simplify the notation we will suppose, throughout this section, to have fixed a time $t \in [0, T]$ and we will write $f, g, \rho, \mathcal{L}$, and w instead of $f(t), g(t), \rho(t), \mathcal{L}(t)$, and $w(t)$, respectively. The objective of this section is to show the existence of a solution to the following minimization problem:

$$\min_{(v,e,p)\in\mathcal{A}(w)} \{\Psi_1(e_{sym}) + \Psi_2(\mathrm{curl}(p)) + \mathcal{H}(p - p_0) - \langle\mathcal{L}, v\rangle\}\,, \tag{4.1}$$

where p_0 is a given admissible plastic strain. To achieve this result we will use the direct method of the Calculus of Variations. We will take a minimizing sequence and show that is bounded in the correct spaces (for this step the safe-load condition will be crucial). Then we will extract a convergent subsequence and show that the limit is a solution to the minimization problem, owing to the lower semicontinuity of the functionals involved. We start to pave the way with some preliminary results.

Lemma 4.1.1. *Let $(H^p_n)_{n\in\mathbb{N}}$ be a sequence of plastic strains such that:*

$$\sup_{n\in\mathbb{N}} \|\,\mathrm{curl}\, H^p_n\|_{L^2} \leq C\,,$$

$$\begin{aligned} H^p_{n,sym} &\overset{*}{\rightharpoonup} H^p_{sym} && \text{in } \mathrm{BV}(\Omega; M^{3\times 3}_{D,sym}), \\ H^p_{n,skew} &\overset{*}{\rightharpoonup} H^p_{skew} && \text{in } \mathscr{M}_b(\Omega; M^{3\times 3}_{skew}). \end{aligned}$$

Then $\operatorname{curl} H^p \in L^2(\Omega; M^{3\times 3})$ *and* $\operatorname{curl} H^p_n \rightharpoonup \operatorname{curl} H^p$ *in* $L^2(\Omega; M^{3\times 3})$*, where* $H^p := H^p_{sym} + H^p_{skew}$.

Proof. Since $L^2(\Omega; M^{3\times 3})$ is reflexive, there is a subsequence $(H^p_{n_k})_{k\in\mathbb{N}}$ and a function $R \in L^2(\Omega; M^{3\times 3})$ such that $\operatorname{curl} H^p_{n_k} \rightharpoonup R$ in $L^2(\Omega; M^{3\times 3})$. Then, for every test function $g \in \mathscr{D}(\Omega; M^{3\times 3})$ we have

$$\begin{aligned} \langle R, g\rangle &= \lim_{k\to\infty} \langle \operatorname{curl} H^p_{n_k}, g\rangle = -\lim_{k\to\infty} \langle H^p_{n_k}, \operatorname{curl}(g^T)^T\rangle = -\langle H^p, \operatorname{curl}(g^T)^T\rangle \\ &= \langle \operatorname{curl} H^p, g\rangle . \end{aligned}$$

By density we deduce that $\operatorname{curl} H^p = R$, so that $\operatorname{curl} H^p \in L^2(\Omega; M^{3\times 3})$. To conclude the proof it is sufficient to show that the extraction of a subsequence is not needed. Let $g \in L^2(\Omega; M^{3\times 3})$ and $\varepsilon > 0$. By density there exists $f \in \mathscr{D}(\Omega; M^{3\times 3})$ such that $\|f - g\|_{L^2} \le \varepsilon$. Since by assumption $(\operatorname{curl}(H^p_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega; M^{3\times 3})$ and $\operatorname{curl}(H^p) \in L^2(\Omega; M^{3\times 3})$, we have

$$\begin{aligned} |\langle \operatorname{curl} H^p_n - \operatorname{curl} H^p, g\rangle| &\le |\langle \operatorname{curl} H^p_n - \operatorname{curl} H^p, g - f\rangle| + |\langle \operatorname{curl} H^p_n - \operatorname{curl} H^p, f\rangle| \\ &\le \|\operatorname{curl} H^p_n - \operatorname{curl} H^p\|_{L^2} \|g - f\|_{L^2} + |\langle \operatorname{curl} H^p_n - \operatorname{curl} H^p, f\rangle| \\ &\le \varepsilon C' + |\langle H^p_n - H^p, \operatorname{curl}(f^T)^T\rangle| . \end{aligned}$$

Passing to the limit we obtain

$$\lim_{n\to\infty} |\langle \operatorname{curl} H^p_n - \operatorname{curl} H^p, g\rangle| \le \varepsilon C'$$

and this concludes the proof, since ε and g are arbitrary. ■

Proposition 4.1.1. *Let* $(w_n) \subset H^1(\Omega; \mathbb{R}^3)$ *be a sequence of boundary values such that* $w_n \rightharpoonup w$ *in* $H^1(\Omega; \mathbb{R}^3)$*. Let* (u_n, H^e_n, H^p_n) *be a sequence of admissible triplets such that*

$$\begin{aligned} &(u_n, H^e_n, H^p_n) \in \mathcal{A}(w_n) && \forall\, n \in \mathbb{N} \\ &u_n \rightharpoonup u && \text{in } W^{1,\frac{3}{2}}(\Omega; \mathbb{R}^3), \\ &H^e_{n,sym} \rightharpoonup H^e_{sym} && \text{in } L^2(\Omega; M^{3\times 3}_{sym}), \\ &\operatorname{curl} H^p_n \rightharpoonup R && \text{in } L^2(\Omega; M^{3\times 3}), \\ &H^p_{n,sym} \overset{*}{\rightharpoonup} H^p_{sym} && \text{in } \mathrm{BV}(\Omega; M^{3\times 3}_{D,sym}), \\ &H^p_{n,skew} \overset{*}{\rightharpoonup} H^p_{skew} && \text{in } \mathscr{M}_b(\Omega; M^{3\times 3}_{skew}). \end{aligned}$$

Then, defining

$$H^p := H^p_{sym} + H^p_{skew}, \tag{4.2}$$

$$H^e_{skew} := \nabla u - H^p - H^e_{sym}, \tag{4.3}$$

$$H^e := H^e_{sym} + H^e_{skew}, \tag{4.4}$$

the triplet (u, H^e, H^p) is admissible for the boundary value w and $R = \operatorname{curl} H^p$.

Proof. Note that properties (3.29) and (3.31) in the definition of $\mathcal{A}(w)$ are automatically granted by the supposed convergences. In view of Lemma 4.1.1 we have $R = \operatorname{curl} H^p$. Finally, by (4.3), condition (3.30) holds true for the triplet (u, H^e, H^p). ∎

The previous result is particularly important since it guarantees the closure of the set $\mathcal{A}(w)$ with respect to the natural convergences. Given a sequence of admissible triplets (u_n, H^e_n, H^p_n) such that all the hypotheses of Proposition 4.1.1 hold true, we will usually denote its limit by (u, H^e, H^p) implying that H^p and H^e are given by (4.2) and (4.4).

We now show a few results for the functional $\mathcal{H}$, that will be used to prove some coercivity property. First of all, we note that

$$((A, B), (C, D))_{\mathcal{H}} = A_{sym} : C_{sym} + \chi A_{skew} : C_{skew} + h^2 B : D$$

is an inner product on the vector space $M^{3\times3} \times M^{3\times3\times3}$. The induced norm is

$$|(A, B)|_{\mathcal{H}} = \sqrt{|A_{sym}|^2 + \chi|A_{skew}|^2 + h^2|B|^2} \tag{4.5}$$

for $(A, B) \in M^{3\times3} \times M^{3\times3\times3}$. We define

$$|(A, B)|^*_{\mathcal{H}} = \sqrt{|A_{sym}|^2 + \frac{1}{\chi}|A_{skew}|^2 + \frac{1}{h^2}|B|^2} \tag{4.6}$$

for $(A, B) \in M^{3\times3} \times M^{3\times3\times3}$. The next proposition states that $|(\cdot,\cdot)|^*_{\mathcal{H}}$ and $|(\cdot,\cdot)|_{\mathcal{H}}$ are dual norms.

Proposition 4.1.2. *The following property holds:*

$$|(A, B)|_{\mathcal{H}} = \sup_{|(C,D)|^*_{\mathcal{H}} \leq 1} \{A : C + B : D\} \tag{4.7}$$

for every $(A, B) \in M^{3\times3} \times M^{3\times3\times3}$.

Proof. Let $(A, B) \in M^{3\times3} \times M^{3\times3\times3}$ and let $(C, D) \in M^{3\times3} \times M^{3\times3\times3}$ be such that $|(C, D)|^*_{\mathcal{H}} \leq 1$. From the Cauchy-Schwarz inequality it follows that

$$\begin{aligned} A : C + B : D &= A_{sym} : C_{sym} + A_{skew} : C_{skew} + B : D \\ &= A_{sym} : C_{sym} + \chi A_{skew} : \left(\frac{1}{\chi}C_{skew}\right) + h^2 B : \left(\frac{1}{h^2}D\right) \\ \leq |(A, B)|_{\mathcal{H}} &\left|\left(C_{sym} + \frac{1}{\chi}C_{skew}, \frac{1}{h^2}D\right)\right|_{\mathcal{H}} = |(A, B)|_{\mathcal{H}}|(C, D)|^*_{\mathcal{H}} \leq |(A, B)|_{\mathcal{H}} \,. \end{aligned}$$

Now let $(C, D) := \dfrac{1}{|(A, B)|_{\mathcal{H}}}(A_{sym} + \chi A_{skew}, h^2 B)$. Clearly,

$$|(C, D)|^*_{\mathcal{H}} = \frac{1}{|(A, B)|_{\mathcal{H}}}|(A_{sym} + \chi A_{skew}, h^2 B)|^*_{\mathcal{H}} = \frac{|(A, B)|_{\mathcal{H}}}{|(A, B)|_{\mathcal{H}}} = 1 \,.$$

By the definition (4.5) of $|(\cdot,\cdot)|_{\mathcal{H}}$, we have

$$A : C + B : D = A_{sym} : C_{sym} + A_{skew} : C_{skew} + B : D$$
$$= \frac{1}{|(A,B)|_{\mathcal{H}}}(A_{sym} : A_{sym} + \chi A_{skew} : A_{skew} + h^2 B : B) = |(A,B)|_{\mathcal{H}}\,.$$

■

With similar arguments one can show that $|(\cdot,\cdot)|^*_{\mathcal{H}}$ is the dual norm of $|(\cdot,\cdot)|_{\mathcal{H}}$, that is

$$|(A,B)|^*_{\mathcal{H}} = \sup_{|(C,D)|_{\mathcal{H}}\leq 1} \{A : C + B : D\} \tag{4.8}$$

for every $(A,B) \in M^{3\times 3} \times M^{3\times 3\times 3}$.

We now extend these results to suitable functional spaces. Let us define the norms

$$\|(F,G)\|_{\mathcal{H}} = \int_\Omega |(F(x),G(x))|_{\mathcal{H}}\; dx \tag{4.9}$$

for $(F,G) \in L^1(\Omega;M^{3\times 3}) \times L^1(\Omega;M^{3\times 3\times 3})$ and

$$\|(F,G)\|^*_{\mathcal{H}} = \operatorname*{ess\,sup}_{x\in\Omega} |(F(x),G(x))|^*_{\mathcal{H}} \tag{4.10}$$

for $(F,G) \in L^\infty(\Omega;M^{3\times 3}) \times L^\infty(\Omega;M^{3\times 3\times 3})$. One can show that

$$\|(F,G)\|_{\mathcal{H}} = \sup_{\|(L,M)\|^*_{\mathcal{H}}\leq 1} \int_\Omega (F : L + G : M)\; dx \tag{4.11}$$

for every $(F,G) \in L^1(\Omega;M^{3\times 3}) \times L^1(\Omega;M^{3\times 3\times 3})$ and

$$\|(L,M)\|^*_{\mathcal{H}} = \sup_{\|(F,G)\|_{\mathcal{H}}\leq 1} \int_\Omega (F : L + G : M)\; dx \tag{4.12}$$

for every $(L,M) \in L^\infty(\Omega;M^{3\times 3}) \times L^\infty(\Omega;M^{3\times 3\times 3})$. These results will be useful in the next proposition since we have, by definition

$$\mathcal{H}_1(H^p) = Y_0\|(H^p, \nabla H^p_{sym})\|_{\mathcal{H}}\,.$$

Proposition 4.1.3. *There are two constants $\alpha,\ \beta > 0$ such that*

$$\mathcal{H}(H^p) - \int_\Omega \rho_D : H^p_{sym}\; dx \geq \alpha\|H^p_{sym}\|_{\mathrm{BV}} + \beta\|H^p_{skew}\|_{L^1} \tag{4.13}$$

for every admissible plastic strain H^p. In addition, we can choose the following values for α and β:

$$\alpha = \min\left\{\frac{M}{2}, \frac{h}{2}\sqrt{\frac{M(4Y_0 - M)}{2}}, hY_0\right\}\,, \tag{4.14}$$

$$\beta = \frac{1}{2}\sqrt{\frac{\chi M(4Y_0 - M)}{2}}\,. \tag{4.15}$$

Proof. Let $0 < M < Y_0$ be the constant given in (3.50) and let us define the sets

$$\mathcal{K} := \Big\{(\tau_1,\tau_2,\tau_3) \in L^\infty(\Omega; M^{3\times3}_{D,sym}) \times L^\infty(\Omega; M^{3\times3}_{skew}) \times L^\infty(\Omega; M^{3\times3\times3}) \;: \\ \|(\tau_1+\tau_2,\tau_3)\|^*_{\mathcal{H}} \le Y_0\Big\}\,,$$

$$\widetilde{\mathcal{K}} = \Bigg\{(\tau_1,\tau_2,\tau_3) \in L^\infty(\Omega; M^{3\times3}_{D,sym}) \times L^\infty(\Omega; M^{3\times3}_{skew}) \times L^\infty(\Omega; M^{3\times3\times3}) \;: \\ \|\tau_1\|_{L^\infty} \le Y_0 - \frac{M}{2},\ \|\tau_2\|_{L^\infty} \le \frac{1}{2}\sqrt{\frac{\chi M(4Y_0 - M)}{2}},\ \|\tau_3\|_{L^\infty} \le \frac{h}{2}\sqrt{\frac{M(4Y_0 - M)}{2}}\Bigg\}\,.$$

By (4.11) we have that, for every $(\tau_1,\tau_2,\tau_3) \in \mathcal{K}$,

$$\mathcal{H}_1(H^p) - \int_\Omega \rho_D : H^p_{sym}\,dx \ge \\ \int_\Omega [(\tau_1 - \rho_D) : H^p_{sym} + \tau_2 : H^p_{skew} + \tau_3 : \nabla^a H^p_{sym}]\,dx\,. \tag{4.16}$$

Note that $\widetilde{\mathcal{K}} \subset \mathcal{K}$ hence (4.16) holds true for every $(\tau_1,\tau_2,\tau_3) \in \widetilde{\mathcal{K}}$. Any function $f \in L^\infty(\Omega, M^{3\times3}_D)$ such that $\|f\|_{L^\infty} \le \frac{M}{2}$ can be written in the form $\tau_1 - \rho_D$ for a suitable $\tau_1 \in L^\infty(\Omega; M^{3\times3}_{D,sym})$ with $\|\tau_1\|_{L^\infty} \le Y_0 - \frac{M}{2}$. Indeed, assumption (3.50) implies that

$$|f + \rho_D| \le Y_0 - \frac{M}{2} \quad \text{a.e in } \Omega\ \forall\ f \in L^\infty(\Omega, M^{3\times3}_D) \text{ such that } \|f\|_{L^\infty} \le \frac{M}{2}\,.$$

Therefore, passing to the supremum on $\widetilde{\mathcal{K}}$ in (4.16) we find that

$$\mathcal{H}^a(H^p) - \int_\Omega \rho_D : H^p_{sym}\,dx \ge \frac{M}{2}\|H^p_{sym}\|_{L^1} \\ + \frac{1}{2}\sqrt{\frac{\chi M(4Y_0 - M)}{2}}\|H^p_{skew}\|_{L^1} + \frac{h}{2}\sqrt{\frac{M(4Y_0 - M)}{2}}\|\nabla^a H^p_{sym}\|_{L^1}\,. \tag{4.17}$$

Adding to (4.17) the remaining term $\mathcal{H}_2(H^p)$, the proof is complete. ∎

Remark 4.1.1. Proposition 4.1.3 gives a coercivity estimate with respect to the norms $\|\cdot\|_{\mathrm{BV}}$ and $\|\cdot\|_{L^1}$. In the proof we have also shown that

$$\mathcal{H}(H^p) - \int_\Omega \rho_D : H^p_{sym}\,dx \ge \frac{M}{2}\|H^p_{sym}\|_{L^1}\,.$$

This estimate will be relevant in Chapter 6 when we will study the asymptotic behavior of solutions as $h, L \to 0$.

We are now ready to prove the main result of this section.

Theorem 4.1.1. *Problem (4.1) has a solution.*

Proof. Let (u_n, H_n^e, H_n^p) be a minimizing sequence for problem (4.1). Note that $(w, \nabla w, 0) \in \mathcal{A}(w)$, so $\mathcal{A}(w) \neq \emptyset$. Hence, there exists a constant $C > 0$ such that

$$\Psi_1(H^e_{n,sym}) + \Psi_2(\mathrm{curl}(H^p_n)) + \mathcal{H}(H^p_n - p_0) - \langle \mathcal{L}, u_n \rangle \le C \quad \forall\, n \in \mathbb{N}\,.$$

By Proposition 3.2.5 we obtain, for every $n \in \mathbb{N}$,

$$\begin{aligned}\Psi_1(H^e_{n,sym}) + \Psi_2(\mathrm{curl}(H^p_n)) + \mathcal{H}(H^p_n - p_0) - \int_\Omega \rho : H^e_{n,sym}\, dx \\ - \int_\Omega \rho_D : H^p_{n,sym}\, dx \le C + \langle \mathcal{L}(t), w \rangle - \int_\Omega \rho(t) : Ew\, dx\,. \end{aligned} \tag{4.18}$$

By (3.44) and the Hölder inequality we infer

$$\begin{aligned}\Psi_1(H^e_{n,sym}) - \int_\Omega \rho : H^e_{n,sym}\, dx &\ge \frac{\alpha_{\mathbb{C}}}{2} \|H^e_{n,sym}\|^2_{L^2} - \|\rho\|_{L^2} \|H^e_{n,sym}\|_{L^2} \\ &\ge \|H^e_{n,sym}\|_{L^2} - \frac{(\|\rho\|_{L^2} + 1)^2}{2\alpha_{\mathbb{C}}}\,. \end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19) with (4.13) we deduce that there exists $C > 0$, independent of n, such that

$$\|H^e_{n,sym}\|_{L^2} + \|\,\mathrm{curl}(H^p_n)\|_{L^2} + \|H^p_{n,sym}\|_{\mathrm{BV}} + \|H^p_{n,skew}\|_{L^1} \le C \quad \forall\, n \in \mathbb{N}\,.$$

By the Korn inequality (2.5) we know that for some constant $C' > 0$

$$\|\nabla u_n\|_{L^{\frac{3}{2}}} \le C'\,.$$

Applying the Poincaré inequality (2.8), the norm $\|u_n\|_{L^{\frac{3}{2}}}$ is uniformly bounded with respect to n. Hence, up to a subsequence, we have:

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } W^{1,\frac{3}{2}}(\Omega; \mathbb{R}^3)\,, \\ H^e_{n,sym} &\rightharpoonup H^e_{sym} && \text{in } L^2(\Omega; M^{3\times 3}_{sym})\,, \\ \mathrm{curl}\, H^p_n &\rightharpoonup R && \text{in } L^2(\Omega; M^{3\times 3})\,, \\ H^p_{n,sym} &\overset{*}{\rightharpoonup} H^p_{sym} && \text{in } \mathrm{BV}(\Omega; M^{3\times 3}_{D,sym})\,, \\ H^p_{n,skew} &\overset{*}{\rightharpoonup} H^p_{skew} && \text{in } \mathscr{M}_b(\Omega; M^{3\times 3}_{skew})\,. \end{aligned}$$

By applying Proposition 4.1.1 we infer that $R = \mathrm{curl}\, H^p$ and $(u, H^e, H^p) \in \mathcal{A}(w)$. By Proposition 3.2.2 it holds

$$\mathcal{H}(H^p - p_0) \le \liminf_{n\to\infty} \mathcal{H}(H^p_n - p_0)\,. \tag{4.20}$$

Moreover, by the continuity and convexity of Ψ_1, Ψ_2 and $\mathcal{L}$ we obtain

$$\begin{aligned}&\Psi_1(H^e_{sym}) + \Psi_2(\mathrm{curl}(H^p)) + \langle \mathcal{L}, u\rangle \\ &\leq \liminf_{n\to\infty}[\Psi_1(H^e_{sym,n}) + \Psi_2(\mathrm{curl}(H^p_n)) + \langle \mathcal{L}, u_n\rangle]\,. \end{aligned} \tag{4.21}$$

Combining (4.20) and (4.21) we deduce that (u, H^e, H^p) is a solution for (4.1). This concludes the proof. ■

4.2 Discretized evolutions

In this section we will discretize the time variable to construct an approximated energetic solution for the Gurtin model. This is a well-established approach to solve rate-independent problems (see [17]). At the end of the section we will prove the global stability condition (3.56) and a discrete energy inequality (3.57) for such discretized solution.

Let us introduce some useful notation. For the rest of the chapter we will denote by $k \in \mathbb{N}$ the discretization parameter and write

$$t^i_k := \frac{i}{k}T \quad \forall\ i = 0, \dots, k\,, \quad k \in \mathbb{N}\,.$$

Abusing this notation we will write $[t^k_k, t^{k+1}_k)$ to denote the degenerate interval $[T, T]$.

We will use the subscript k to denote the right-continuous piecewise constant interpolation of a function defined at the discrete times t^i_k. As an example consider

$$w_k(t) = w(t^i_k) \quad \text{if } t \in [t^i_k, t^{i+1}_k)\,.$$

We fix as initial condition an admissible triplet (u_0, H^e_0, H^p_0) for the boundary value $w(0)$ and we assume that, for this triplet, the global stability condition (3.56) holds true, that is,

$$\begin{aligned}\Psi_1(H^e_{0,sym}) + \Psi_2(\mathrm{curl}(H^p_0)) - \langle \mathcal{L}(0), u_0\rangle &\leq \Psi_1(e_{sym}) + \Psi_2(\mathrm{curl}(p)) \\ &-\langle \mathcal{L}(0), v\rangle + \mathcal{H}(p - H^p_0) \quad \forall\ (v, e, p) \in \mathcal{A}(w(0))\,. \end{aligned} \tag{4.22}$$

We construct an approximate energetic solution for the Gurtin model inductively. Let

$$(u_{0,k}, H^e_{0,k}, H^p_{0,k}) := (u_0, H^e_0, H^p_0) \quad \forall\ k \in \mathbb{N}\,.$$

For every $i = 1, \dots, k$ let $(u_{i,k}, H^e_{i,k}, H^p_{i,k}) \in \mathcal{A}(w(t^i_k))$ be a solution to problem (4.1) with $p_0 = H^p_{i-1,k}$. Now let us define u_k, H^e_k and H^p_k as the piecewise constant interpolants given by

$$\begin{aligned} u_k(t) &:= u_{i,k} \quad \forall\ t \in [t^i_k, t^{i+1}_k)\,, \\ H^e_k(t) &:= H^e_{i,k} \quad \forall\ t \in [t^i_k, t^{i+1}_k)\,, \\ H^p_k(t) &:= H^p_{i,k} \quad \forall\ t \in [t^i_k, t^{i+1}_k)\,, \end{aligned}$$

for every $i \in \{0, \dots, k\}$. In a similar fashion we define

$$\mathcal{E}_k(t) := \Psi_1(H^e_{k,sym}(t)) + \Psi_2(\mathrm{curl}(H^p_k(t))) - \langle \mathcal{L}_k(t), u_k(t) \rangle ,$$

where $\mathcal{L}_k(t)$ is defined as in (3.27) with $f = f_k$ and $g = g_k$. By construction we have

$$(u_k(t), H^e_k(t), H^p_k(t)) \in \mathcal{A}(w_k(t)) \quad \forall\, t \in [0,T]\ \forall\, k \in \mathbb{N} . \tag{4.23}$$

We are ready to prove the main results of the section.

Proposition 4.2.1. *The following global stability condition holds true: for every $t \in [0,T]$ and every $k \in \mathbb{N}$*

$$\begin{aligned}\mathcal{E}_k(t) \le \Psi_1(e_{sym}) + \Psi_2(\mathrm{curl}(p)) - \langle \mathcal{L}_k(t), v \rangle + \mathcal{H}(p - H^p_k(t)) \\ \forall\, (v,e,p) \in \mathcal{A}(w_k(t)) .\end{aligned} \tag{4.24}$$

Proof. If $t \in [0, t^1_k)$, the claim is true by assumption (4.22). Let $i \in \{1, \dots, k\}$ and $t \in [t^i_k, t^{i+1}_k)$. By definition of the approximate solutions we have

$$\begin{aligned}&\Psi_1(H^e_{k,sym}(t)) + \Psi_2(\mathrm{curl}(H^p_k(t))) - \langle \mathcal{L}_k(t), u_k(t) \rangle + \mathcal{H}(H^p_k(t) - H^p_k(t^{i-1}_k)) \\ &\le \Psi_1(e_{sym}) + \Psi_2(\mathrm{curl}(p)) - \langle \mathcal{L}_k(t), v \rangle + \mathcal{H}(p - H^p_k(t^{i-1}_k)) \\ &\quad \forall\, (v,e,p) \in \mathcal{A}(w_k(t)) .\end{aligned}$$

By the subadditivity of $\mathcal{H}$ the last term can be estimated in the following way:

$$\mathcal{H}(p - H^p_k(t^{i-1}_k)) \le \mathcal{H}(p - H^p_k(t)) + \mathcal{H}(H^p_k(t) - H^p_k(t^{i-1}_k)) ,$$

concluding the proof. ∎

Proposition 4.2.2. *The following discrete energy inequality holds: for every $k \in \mathbb{N}$ and $j \in \{0, \dots, k\}$*

$$\begin{aligned}\mathcal{E}_k(t) + \mathcal{V}_{\mathcal{H}}(H^p_k; 0, t) \le \mathcal{E}(0) + \int_0^{t^j_k} \int_\Omega \mathbb{C} H^e_{k,sym}(\tau) : E\dot{w}(\tau)\, dx d\tau \\ - \int_0^{t^j_k} \langle \dot{\mathcal{L}}(\tau), u_k(\tau) \rangle\, d\tau - \int_0^{t^j_k} \langle \mathcal{L}_k(\tau), \dot{w}(\tau) \rangle\, d\tau + \varepsilon_k \quad \forall\, t \in [t^j_k, t^{j+1}_k) ,\end{aligned} \tag{4.25}$$

where $\varepsilon_k \to 0$ as $k \to +\infty$.

Proof. The claim is trivially true if $t \in [0, t^1_k)$. Let us suppose $j \in \{1, \dots, k\}$ and $t \in [t^j_k, t^{j+1}_k)$. Let $i \in \{1, \dots, j\}$. Note that the triplet

$$(u_k(t^{i-1}_k) + w_k(t^i_k) - w_k(t^{i-1}_k), H^e_k(t^{i-1}_k) + \nabla w_k(t^i_k) - \nabla w_k(t^{i-1}_k), H^p_k(t^{i-1}_k))$$

is admissible for the boundary value $w_k(t^i_k)$. Hence, by definition of the approximate solutions, we have

$$\begin{aligned}&\Psi_1(H^e_{k,sym}(t^i_k)) + \Psi_2(\mathrm{curl}(H^p_k(t^i_k))) - \langle \mathcal{L}_k(t^i_k), u_k(t^i_k) \rangle \\ &+ \mathcal{H}(H^p_k(t^i_k) - H^p_k(t^{i-1}_k)) \le \Psi_1(H^e_{k,sym}(t^{i-1}_k) + Ew_k(t^i_k) - Ew_k(t^{i-1}_k))\end{aligned}$$

$$+\Psi_2(\mathrm{curl}(H_k^p(t_k^{i-1}))) - \langle \mathcal{L}_k(t_k^i), u_k(t_k^{i-1}) + w_k(t_k^i) - w_k(t_k^{i-1})\rangle \,. \tag{4.26}$$

We focus now on the right-hand side of (4.26) analyzing one term at a time. The first one can be decomposed as follows:

$$\begin{aligned}
&\Psi_1(H_{k,sym}^e(t_k^{i-1}) + Ew_k(t_k^i) - Ew_k(t_k^{i-1})) = \Psi_1(H_{k,sym}^e(t_k^{i-1})) \\
&+ \Psi_1(Ew_k(t_k^i) - Ew_k(t_k^{i-1})) + \int_\Omega \mathbb{C}H_{k,sym}^e(t_k^{i-1}) : (Ew_k(t_k^i) - Ew_k(t_k^{i-1}))\, dx \,.
\end{aligned} \tag{4.27}$$

Moreover, the last term in (4.27) can be rewritten as follows:

$$\begin{aligned}
&\int_\Omega \mathbb{C}H_{k,sym}^e(t_k^{i-1}) : (Ew_k(t_k^i) - Ew_k(t_k^{i-1}))\, dx \\
&= \int_\Omega \int_{t_k^{i-1}}^{t_k^i} \frac{d}{d\tau}[\mathbb{C}H_{k,sym}^e(t_k^{i-1}) : Ew(\tau)]\, d\tau dx \\
&= \int_\Omega \int_{t_k^{i-1}}^{t_k^i} \mathbb{C}H_{k,sym}^e(t_k^{i-1}) : E\dot{w}(\tau)\, d\tau dx = \int_{t_k^{i-1}}^{t_k^i} \int_\Omega \mathbb{C}H_{k,sym}^e(\tau) : E\dot{w}(\tau)\, d\tau dx \,.
\end{aligned} \tag{4.28}$$

The last term at the right-hand side of (4.26) can be manipulated to get

$$\begin{aligned}
&\langle \mathcal{L}_k(t_k^i), u_k(t_k^{i-1}) + w_k(t_k^i) - w_k(t_k^{i-1})\rangle \\
&= \langle \mathcal{L}_k(t_k^i) - \mathcal{L}_k(t_k^{i-1}) + \mathcal{L}_k(t_k^{i-1}), u_k(t_k^{i-1}) + w_k(t_k^i) - w_k(t_k^{i-1})\rangle \\
&= \langle \mathcal{L}_k(t_k^{i-1}), u_k(t_k^{i-1})\rangle + \langle \mathcal{L}_k(t_k^i) - \mathcal{L}_k(t_k^{i-1}), w_k(t_k^i) - w_k(t_k^{i-1})\rangle \\
&+ \langle \mathcal{L}_k(t_k^{i-1}), w_k(t_k^i) - w_k(t_k^{i-1})\rangle + \langle \mathcal{L}_k(t_k^i) - \mathcal{L}_k(t_k^{i-1}), u_k(t_k^{i-1})\rangle \,.
\end{aligned} \tag{4.29}$$

The last two terms in (4.29) can be rewritten in integral form:

$$\begin{aligned}
\langle \mathcal{L}_k(t_k^{i-1}), w_k(t_k^i) - w_k(t_k^{i-1})\rangle &= \int_{t_k^{i-1}}^{t_k^i} \langle \mathcal{L}_k(t_k^{i-1}), \dot{w}(t)\rangle\, d\tau \\
&= \int_{t_k^{i-1}}^{t_k^i} \langle \mathcal{L}_k(\tau), \dot{w}(\tau)\rangle\, d\tau \,,
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\langle \mathcal{L}_k(t_k^i) - \mathcal{L}_k(t_k^{i-1}), u_k(t_k^{i-1})\rangle &= \int_{t_k^{i-1}}^{t_k^i} \langle \dot{\mathcal{L}}(\tau), u_k(t_k^{i-1})\rangle\, d\tau \\
&= \int_{t_k^{i-1}}^{t_k^i} \langle \dot{\mathcal{L}}(\tau), u_k(\tau)\rangle\, d\tau \,.
\end{aligned} \tag{4.31}$$

Lastly, let us define

$$\begin{aligned}
\delta_{k,i} :=& \Psi_1(Ew_k(t_k^i) - Ew_k(t_k^{i-1})) \\
&+ |\langle \mathcal{L}_k(t_k^i) - \mathcal{L}_k(t_k^{i-1}), w_k(t_k^i) - w_k(t_k^{i-1})\rangle| \,.
\end{aligned} \tag{4.32}$$

Combining the identities (4.27)-(4.31) and definition (4.32) into inequality (4.26) we conclude that

$$
\begin{aligned}
&\mathcal{E}_k(t_k^i)+\mathcal{H}(H_k^p(t_k^i)-H_k^p(t_k^{i-1}))\le \mathcal{E}_k(t_k^{i-1})\\
&+\int_{t_k^{i-1}}^{t_k^i}\int_\Omega \mathbb{C}H^e_{k,sym}(\tau):E\dot w(\tau)\;d\tau dx-\int_{t_k^{i-1}}^{t_k^i}\langle \mathcal{L}_k(\tau),\dot w(t)\rangle\;d\tau\\
&-\int_{t_k^{i-1}}^{t_k^i}\langle \dot{\mathcal{L}}(\tau),u_k(\tau)\rangle\;d\tau+\delta_{k,i}\,.
\end{aligned}
\tag{4.33}
$$

Summing up (4.33) for $i=1,\dots,j$ we infer

$$
\begin{aligned}
&\mathcal{E}_k(t_k^j)+\mathcal{V}_{\mathcal{H}}(H_k^p;0,t)\le \mathcal{E}_k(0)+\int_0^{t_k^j}\int_\Omega \mathbb{C}H^e_{k,sym}(\tau):E\dot w_k(\tau)\;d\tau dx\\
&-\int_0^{t_k^j}\langle \mathcal{L}_k(\tau),\dot w_k(\tau)\rangle\;d\tau-\int_0^{t_k^j}\langle \dot{\mathcal{L}}_k(\tau),u_k(\tau)\rangle\;d\tau+\sum_{i=1}^k\delta_{k,i}\,.
\end{aligned}
$$

Note that by definition we have $\mathcal{E}_k(t)=\mathcal{E}_k(t_k^j)$ and $\mathcal{E}_k(0)=\mathcal{E}(0)$. To conclude the proof we just need to prove that $\varepsilon_k:=\sum_{i=1}^k\delta_{k,i}\to 0$, as $k\to\infty$. First of all, by the Jensen inequality and the coercivity estimate (3.44), we deduce that

$$
\begin{aligned}
&\Psi_1(Ew_k(t_k^i)-Ew_k(t_k^{i-1}))=\Psi_1\left(\int_{t_k^{i-1}}^{t_k^i}E\dot w(\tau)\;d\tau\right)\\
\le&\frac{\beta_{\mathbb{C}}}{2}\int_\Omega\left|\int_{t_k^{i-1}}^{t_k^i}E\dot w(\tau)\;d\tau\right|^2\;dx=\frac{\beta_{\mathbb{C}}}{2}\int_\Omega\frac{T^2}{k^2}\left|\frac{k}{T}\int_{t_k^{i-1}}^{t_k^i}E\dot w(\tau)\;d\tau\right|^2\;dx\\
\le&\frac{\beta_{\mathbb{C}}T}{2k}\int_{t_k^{i-1}}^{t_k^i}\|E\dot w(\tau)\|^2_{L^2}\;d\tau\,.
\end{aligned}
\tag{4.34}
$$

Since $\mathcal{L}(t)$ is a linear and continuous operator for every $t\in[0,T]$ and, in addition, $\mathcal{L}_k(t_k^i)=\mathcal{L}(t_k^i)$ for every $i=1,\dots,k$, we infer that

$$
\begin{aligned}
&|\langle \mathcal{L}_k(t_k^i)-\mathcal{L}_k(t_k^{i-1}),w_k(t_k^i)-w_k(t_k^{i-1})\rangle|=\left|\langle \mathcal{L}(t_k^i)-\mathcal{L}(t_k^{i-1}),\int_{t_k^{i-1}}^{t_k^i}\dot w(\tau)\;d\tau\rangle\right|\\
\le&\|\mathcal{L}(t_k^i)-\mathcal{L}(t_k^{i-1})\|_{W^{-1,3}}\left\|\int_{t_k^{i-1}}^{t_k^i}\dot w(\tau)\;d\tau\right\|_{W^{1,\frac32}}\\
\le&\sup_{i=1,\dots,k}\|\mathcal{L}(t_k^i)-\mathcal{L}(t_k^{i-1})\|_{W^{-1,3}}\int_{t_k^{i-1}}^{t_k^i}\|\dot w(\tau)\|_{W^{1,\frac32}}\;d\tau\,.
\end{aligned}
\tag{4.35}
$$

Combining (4.34) and (4.35) we deduce that

$$
\begin{aligned}
\varepsilon_k\le&\frac{\beta_{\mathbb{C}}T}{2k}\int_0^T\|E\dot w(\tau)\|^2_{L^2}\;d\tau\\
&+\sup_{i=1,\dots,k}\|\mathcal{L}(t_k^i)-\mathcal{L}(t_k^{i-1})\|_{W^{-1,3}}\int_0^T\|\dot w(\tau)\|_{W^{1,\frac32}}\;d\tau\,.
\end{aligned}
$$

Finally, recalling that $\mathcal{L}$ is absolutely continuous with respect to time with values in the space $W^{-1,3}(\Omega;\mathbb{R}^3)$, we conclude that $\varepsilon_k \to 0$ as $k \to \infty$. ∎

4.3 The existence result

To pass to the limit in the global stability condition (4.24) and the discrete energy inequality (4.25) we need to establish some compactness for the approximate solutions. To this aim, in the next proposition we prove some uniform bound with respect to the discretization parameter k of the approximate solutions.

Proposition 4.3.1. *There are two constants $C_1, C_2 > 0$, depending only on $T, \alpha_{\mathbb{C}}, \beta_{\mathbb{C}}, Ew, E\dot{w}, \rho, \dot{\rho}$, and the initial triplet (u_0, H_0^e, H_0^p), such that for every $k \in \mathbb{N}$ and $t \in [0,T]$ the following estimates hold:*

$$\|H^e_{k,sym}(t)\|_{L^2} \le C_1\,, \tag{4.36}$$

$$\frac{\mu L^2}{2}\|\operatorname{curl}(H^p_k(t))\|^2_{L^2} + \alpha\mathcal{V}(H^p_{k,sym};0,t) + \beta\mathcal{V}(H^p_{k,skew};0,t) \le C_2\,, \tag{4.37}$$

where α and β are the constants in (4.14) and (4.15). Moreover, there exists a constant $C_3 > 0$, depending only on $T, \alpha_{\mathbb{C}}, \beta_{\mathbb{C}}, Ew, E\dot{w}, \rho, \dot{\rho}, C_P, K$, and the initial triplet (u_0, H_0^e, H_0^p), such that

$$\|u_k(t)\|_{W^{1,\frac{3}{2}}} \le C_3 + \frac{1}{\alpha}C_3 \quad \forall\, k \in \mathbb{N}\ \forall\, t \in [0,T]\,. \tag{4.38}$$

where C_P and K are given in Theorem 2.2.2 and Theorem 2.2.4.

Proof. Let $i \in \{1,\dots,k\}$. We can rewrite (4.26) using identity (3.51) to deduce

$$\begin{aligned}
&\Psi_1(H^e_{k,sym}(t^i_k)) + \Psi_2(\operatorname{curl}(H^p_k(t^i_k))) - \int_\Omega \rho(t^i_k) : H^e_{k,sym}(t^i_k)\,dx \\
&- \int_\Omega \rho_D(t^i_k) : H^p_{k,sym}(t^i_k)\,dx + \mathcal{H}(H^p_k(t^i_k) - H^p_k(t^{i-1}_k)) \\
&\le \Psi_1(H^e_{k,sym}(t^{i-1}_k) + Ew_k(t^i_k) - Ew_k(t^{i-1}_k)) + \Psi_2(\operatorname{curl}(H^p_k(t^{i-1}_k))) \\
&- \int_\Omega \rho(t^i_k) : (H^e_{k,sym}(t^{i-1}_k) + Ew_k(t^i_k) - Ew_k(t^{i-1}_k))\,dx \\
&- \int_\Omega \rho_D(t^i_k) : H^p_{k,sym}(t^{i-1}_k)\,dx\,.
\end{aligned} \tag{4.39}$$

Let us define

$$\widetilde{\delta}_{k,i} := \Psi_1(Ew_k(t^i_k) - Ew_k(t^{i-1}_k))\,.$$

By (4.27), (4.28), and (4.39) we obtain

$$\begin{aligned}
&\Psi_1(H^e_{k,sym}(t^i_k)) - \int_\Omega \rho(t^i_k) : (H^e_{k,sym}(t^i_k) - Ew_k(t^i_k))\,dx \\
&+ \Psi_2(\operatorname{curl}(H^p_k(t^i_k))) + \mathcal{H}(H^p_k(t^i_k) - H^p_k(t^{i-1}_k)) \\
&- \int_\Omega \rho_D(t^i_k) : (H^p_{k,sym}(t^i_k) - H^p_{k,sym}(t^{i-1}_k))\,dx \le \Psi_1(H^e_{k,sym}(t^{i-1}_k))
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \rho(t_k^{i-1}) : (H^e_{k,sym}(t_k^{i-1}) - Ew_k(t_k^{i-1}))\ dx + \Psi_2(\mathrm{curl}(H_k^p(t_k^{i-1})))\\
&-\int_\Omega (\rho(t_k^i) - \rho(t_k^{i-1})) : (H^e_{k,sym}(t_k^{i-1}) - Ew_k(t_k^{i-1}))\ dx\\
&+\int_{t_k^{i-1}}^{t_k^i}\int_\Omega \mathbb{C}H^e_{k,sym}(\tau) : E\dot{w}(\tau)\ d\tau dx + \widetilde{\delta}_{k.i}\,. \qquad (4.40)
\end{aligned}
$$

Note that we can rewrite the second to last term as follows:

$$
\begin{aligned}
&\int_\Omega (\rho(t_k^i) - \rho(t_k^{i-1})) : (H^e_{k,sym}(t_k^{i-1}) - Ew_k(t_k^{i-1}))\ dx\\
=&\int_{t_k^{i-1}}^{t_k^i}\int_\Omega \dot{\rho}(\tau) : (H^e_{k,sym}(\tau) - Ew_k(\tau))\ dx d\tau\,. \qquad (4.41)
\end{aligned}
$$

Let $j \in \{1, \ldots, k\}$. Combining (4.40) and (4.41) and summing up for $i = 1, \ldots, j$ we deduce

$$
\begin{aligned}
&\Psi_1(H^e_{k,sym}(t_k^j)) - \int_\Omega \rho(t_k^j) : (H^e_{k,sym}(t_k^j) - Ew_k(t_k^j))\ dx\\
&+\Psi_2(\mathrm{curl}(H_k^p(t_k^j))) + \sum_{i=1}^j \Big[\mathcal{H}(H_k^p(t_k^i) - H_k^p(t_k^{i-1}))\\
&-\int_\Omega \rho_D(t_k^i) : (H^p_{k,sym}(t_k^i) - H^p_{k,sym}(t_k^{i-1}))\ dx\Big] \le \Psi_1(H^e_{0,sym})\\
&-\int_\Omega \rho(0) : (H^e_{0,sym} - Ew(0))\ dx + \Psi_2(\mathrm{curl}(H_0^p))\\
&-\int_0^{t_k^j}\int_\Omega \dot{\rho}(\tau) : (H^e_{k,sym}(\tau) - Ew_k(\tau))\ dx d\tau\\
&+\int_0^{t_k^j}\int_\Omega \mathbb{C}H^e_{k,sym}(\tau) : E\dot{w}(\tau)\ d\tau dx + \sum_{i=1}^k \widetilde{\delta}_{k.i}\,. \qquad (4.42)
\end{aligned}
$$

In the proof of Proposition 4.2.2 (see equation (4.34)) we have already shown that $\sum_{i=1}^k \widetilde{\delta}_{k,i}$ is bounded by a constant $C > 0$ that depends only on $T, \beta_{\mathbb{C}}$, and $E\dot{w}$. Moreover, by Proposition 4.1.3 we have the lower bound

$$
\begin{aligned}
&\sum_{i=1}^j \Big[\mathcal{H}(H_k^p(t_k^i) - H_k^p(t_k^{i-1})) - \int_\Omega \rho_D(t_k^i) : (H^p_{k,sym}(t_k^i) - H^p_{k,sym}(t_k^{i-1}))\ dx\Big]\\
\ge&\sum_{i=1}^j \Big[\alpha\|H^p_{k,sym}(t_k^i) - H^p_{k,sym}(t_k^{i-1})\|_{\mathrm{BV}} + \beta\|H^p_{k,skew}(t_k^i) - H^p_{k,skew}(t_k^{i-1})\|_{L^1}\Big]\\
=&\alpha\mathcal{V}(H^p_{k,sym}; 0, t) + \beta\mathcal{V}(H^p_{k,skew}; 0, t) \quad \forall\ t \in [t_k^j, t_k^{j+1})\,. \qquad (4.43)
\end{aligned}
$$

Combining (4.42) and (4.43) we infer that there exists a new constant $C > 0$, depending only on $T, \beta_{\mathbb{C}}, E\dot{w}, \rho, Ew$, and the initial triplet (u_0, H_0^e, H_0^p), such that

$$
\Psi_1(H^e_{k,sym}(t))\ dx - \int_\Omega \rho(t_k^j) : (H^e_{k,sym}(t) - Ew(t_k^j))\ dx
$$

$$
\begin{aligned}
&+ \Psi_2(\mathrm{curl}(H^p_k(t))) + \alpha\mathcal{V}(H^p_{k,sym};0,t) + \beta\mathcal{V}(H^p_{k,skew};0,t)\\
&\leq C - \int_0^{t^j_k}\int_\Omega \dot\rho(\tau) : (H^e_{k,sym}(\tau) - Ew_k(\tau))\, dx d\tau\\
&+ \int_0^{t^j_k}\int_\Omega \mathbb{C}H^e_{k,sym}(\tau) : E\dot w(\tau)\, d\tau dx \quad \forall\, t \in [t^j_k, t^{j+1}_k)\,. \qquad (4.44)
\end{aligned}
$$

Using the Hölder inequality and (3.44) we deduce from (4.44) that

$$
\begin{aligned}
&\frac{\alpha_{\mathbb{C}}}{2}\|H^e_{k,sym}(t)\|^2_{L^2} - \|\rho(t^j_k)\|_{L^2}\|H^e_{k,sym}(t) - Ew(t^j_k)\|_{L^2}\\
&+\frac{\mu L^2}{2}\|\mathrm{curl}(H^p(t))\|^2_{L^2} + \alpha\mathcal{V}(H^p_{k,sym};0,t) + \beta\mathcal{V}(H^p_{k,skew};0,t)\\
&\leq C + \int_0^t \|\dot\rho(\tau)\|_{L^2}\|H^e_{k,sym}(\tau) - Ew_k(\tau)\|_{L^2}\, d\tau\\
&+\int_0^t \|\mathbb{C}H^e_{k,sym}(\tau)\|_{L^2}\|E\dot w(\tau)\|_{L^2}\, d\tau \quad \forall\, t \in [t^j_k, t^{j+1}_k)\,. \qquad (4.45)
\end{aligned}
$$

Since the first term is quadratic, there exists a constant $C' > 0$, depending only on $\alpha_{\mathbb{C}}$, ρ, Ew, such that

$$
\begin{aligned}
&\frac{\alpha_{\mathbb{C}}}{2}\|H^e_{k,sym}(t)\|^2_{L^2} - \|\rho(t^j_k)\|_{L^2}\|H^e_{k,sym}(t) - Ew(t^j_k)\|_{L^2}\\
&\geq C'\|H^e_{k,sym}(t)\|^2_{L^2} - C' \qquad (4.46)
\end{aligned}
$$

We now focus on the terms that contain $H^e_{k,sym}$ in (4.45). By (4.46) we obtain

$$
\|H^e_{k,sym}(t)\|^2_{L^2} \leq C + C\int_0^t (\|\dot\rho(\tau)\|_{L^2} + \beta_{\mathbb{C}}\|E\dot w(\tau)\|_{L^2})\|H^e_{k,sym}(\tau)\|_{L^2}\, d\tau \quad (4.47)
$$

for every $t \in [t^1_k, T]$ where $C > 0$ is a new constant, depending only on $T, \alpha_{\mathbb{C}}, \beta_{\mathbb{C}}, E\dot w, \rho, Ew$ and the initial triplet (u_0, H^e_0, H^p_0). Note that up to a change of the constant C (but not of its dependence), (4.47) holds true also for $t \in [0, t^1_k)$. By the Lemma 5.3 in [15], since $\dot\rho\,, E\dot w \in L^1(0,T;L^2(\Omega;M^{3\times3}_{sym}))$, (4.36) holds. By (4.36) and (4.45)-(4.46) claim (4.37) immediately follows. Applying Theorem 2.2.2 and Theorem 2.2.4 we obtain (4.38). ■

Remark 4.3.1. Proposition 4.3.1 provides some bounds for the approximate solutions. The same property holds for any energetic solution to the Gurtin model and it can be proved using the same arguments with some minor changes. Moreover, following the same lines of the previous proof and applying Remark 4.1.1, one can show that

$$
\frac{M}{2}\mathcal{V}(H^p_{sym};0,t;L^1) \leq C_2 \quad \forall\, t \in [0,T]\,, \qquad (4.48)
$$

where $\mathcal{V}(H^p_{sym};0,t;L^1)$ is the variation on $[0,t]$ of

$$
H^p_{sym} : [0,T] \to L^1(\Omega; M^{3\times3}_{D,sym})\,.
$$

Finally, C_1 and C_2 have the following form:

$$C_i = C_i' |\psi(H_0^e, \mathrm{curl}(H_0^p))| + C_i' \|H_{0,sym}^e\|_{L^2} + C_i' \,, \tag{4.49}$$

where C_1' and C_2' do not depend on the initial triplet. This remark will be useful in Chapter 6 where we will need to exploit the exact dependence of the estimates (4.36)-(4.38) on χ, h, L and the initial triplet.

We are finally in a position to prove the existence of a solution for the Gurtin model.

Theorem 4.3.1. *Let $(u_0, H_0^e, H_0^p) \in \mathcal{A}(w(0))$ be an admissible triplet such that (4.22) holds true. Then there exists an energetic solution for the Gurtin model with initial datum (u_0, H_0^e, H_0^p).*

Proof. Let $t \mapsto (u_k(t), H_k^e(t), H_k^p(t))$ be an approximate solution, defined as in the previous sections, starting from the initial value (u_0, H_0^e, H_0^p). We will show in this proof that it is possible to pass to the limit in (4.24) and (4.25) as $k \to \infty$, at least along a subsequence, to obtain an energetic solution for the Gurtin model. By Proposition 4.3.1, in view of Theorem 2.1.2, there exists a subsequence (k_n) and two maps with bounded variation

$$\begin{aligned} H_{sym}^p &: [0,T] \to \mathrm{BV}(\Omega; M_{D,sym}^{3\times3}) \,, \\ H_{skew}^p &: [0,T] \to \mathscr{M}_b(\Omega; M_{skew}^{3\times3}) \,, \end{aligned}$$

such that for every $t \in [0,T]$

$$H_{k_n,sym}^p(t) \overset{*}{\rightharpoonup} H_{sym}^p(t) \quad \text{in } \mathrm{BV}(\Omega; M_{D,sym}^{3\times3}) \,, \tag{4.50}$$

$$H_{k_n,skew}^p(t) \overset{*}{\rightharpoonup} H_{skew}^p(t) \quad \text{in } \mathscr{M}_b(\Omega; M_{skew}^{3\times3}) \,. \tag{4.51}$$

Let $H^p(t) := H_{sym}^p(t) + H_{skew}^p(t)$. By Lemma 4.1.1 and (4.37), without extracting a further subsequence, we have

$$\mathrm{curl}(H_{k_n}^p(t)) \rightharpoonup \mathrm{curl}(H^p(t)) \quad \text{in } L^2(\Omega; M^{3\times3}) \quad \forall\, t \in [0,T] \,. \tag{4.52}$$

Applying Proposition 4.3.1 once again we deduce that, for every $t \in [0,T]$, there is a subsequence $(k_{n_{m(t)}})$, which a priori depends on t, and there exist maps $u(t) \in W^{1,\frac{3}{2}}(\Omega; \mathbb{R}^3)$ and $H_{sym}^e(t) \in L^2(\Omega; M_{sym}^{3\times3})$ such that

$$u_{k_{n_{m(t)}}}(t) \rightharpoonup u(t) \qquad \text{in } W^{1,\frac{3}{2}}(\Omega; \mathbb{R}^3) \,, \tag{4.53}$$

$$H_{k_{n_{m(t)}},sym}^e(t) \rightharpoonup H_{sym}^e(t) \quad \text{in } L^2(\Omega; M_{sym}^{3\times3}) \,. \tag{4.54}$$

Since w is absolutely continuous, by (4.23) and Proposition 4.1.1 we have

$$(u(t), H^e(t), H^p(t)) \in \mathcal{A}(w(t)) \quad \forall\, t \in [0,T] \,. \tag{4.55}$$

In particular, by Proposition 3.2.1 H_{skew}^p is absolutely continuous with respect to the Lebesgue measure, thus it has bounded variation as a map with values in $L^1(\Omega; M^{3\times 3}_{skew})$. We first show the global stability condition (3.56) for the map

$$t \mapsto (u(t), H^e(t), H^p(t)) .$$

Let $t \in [0,T]$ and let $(v,e,p) \in \mathcal{A}(w(t))$ be an admissible triplet. In order to simplify the notation let h be any index of the subsequence $k_{n_{m(t)}}$. By (4.23) the triplet

$$(v - u(t) + u_h(t), e - H^e(t) + H^e_h(t), p - H^p(t) + H^p_h(t))$$

is admissible for the boundary value $w_h(t)$. Hence we can apply the global stability condition (4.24) to obtain

$$\begin{aligned}&\Psi_1(H^e_{h,sym}(t)) + \Psi_2(\mathrm{curl}(H^p_h(t))) - \langle \mathcal{L}_h(t), u_h(t)\rangle \leq \\ &\Psi_1(e_{sym} - H^e_{sym}(t) + H^e_{h,sym}(t)) + \Psi_2(\mathrm{curl}(p - H^p(t) + H^p_h(t))) \\ &- \langle \mathcal{L}_h(t), v - u(t) + u_h(t)\rangle + \mathcal{H}(p - H^p(t)) ,\end{aligned}$$

or equivalently

$$\begin{aligned}&0 \leq \Psi_1(e_{sym} - H^e_{sym}(t)) + \int_\Omega \mathbb{C}(e_{sym} - H^e_{sym}(t)) : H^e_{h,sym}(t)\ dx \\ &+\Psi_2(\mathrm{curl}(p - H^p(t))) + \mu L^2 \int_\Omega \mathrm{curl}(p - H^p(t)) : \mathrm{curl}(H^p_h(t))\ dx \\ &-\langle \mathcal{L}_h(t), v - u(t)\rangle + \mathcal{H}(p - H^p(t)) . \end{aligned} \tag{4.56}$$

By the convergences (4.52) and (4.54) and the absolute continuity of $\mathcal{L}$, we can pass to the limit in (4.56) and obtain

$$\begin{aligned}&0 \leq \Psi_1(e_{sym} - H^e_{sym}(t))\ dx + \int_\Omega \mathbb{C}(e_{sym} - H^e_{sym}(t)) : H^e_{sym}(t)\ dx \\ &+\Psi_2(\mathrm{curl}(p - H^p(t))) + \mu L^2 \int_\Omega \mathrm{curl}(p - H^p(t)) : \mathrm{curl}(H^p(t))\ dx \\ &-\langle \mathcal{L}(t), v - u(t)\rangle + \mathcal{H}(p - H^p(t)) . \end{aligned} \tag{4.57}$$

Adding to both members in (4.57) the quantity

$$\Psi_1(H^e_{sym}(t)) + \Psi_2(\mathrm{curl}(H^p(t))) - \langle \mathcal{L}(t), u(t)\rangle$$

we conclude that the global stability condition (3.56) holds. We now prove that the convergences (4.53) and (4.54) hold along the whole subsequence k_n, for every $t \in [0,T]$. Let us fix $t \in [0,T]$. In view of the global stability condition (3.56) the pair $(u(t), H^e(t))$ minimizes the convex functional

$$(v,e) \mapsto \Psi_1(e_{sym}) - \langle \mathcal{L}(t), v\rangle \tag{4.58}$$

over the convex set $\{(v,e) \mid (v,e,H^p(t)) \in \mathcal{A}(w(t))\}$. The functional (4.58) is strictly convex in the second variable. Moreover, if v_1, v_2 are such that the triplets $(v_1, e, H^p(t)), (v_2, e, H^p(t))$ are both admissible for the boundary value

$w(t)$, then

$$\nabla(v_1 - v_2) = 0\,,$$

hence $v_1 = v_2$ since they have the same boundary value on Γ_D. Therefore, the minimizer of (4.58) is unique. By the Urysohn property we conclude that

$$u_{k_n}(t) \rightharpoonup u(t) \qquad \text{in } W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3)\,, \tag{4.59}$$

$$H^e_{k_n,sym}(t) \rightharpoonup H^e_{sym}(t) \quad \text{in } L^2(\Omega; M^{3\times 3}_{sym})\,. \tag{4.60}$$

It remains to prove the energy balance. To simplify the notation let us denote by (k) the subsequence $(k_n)_n$. Let us fix $t \in [0,T]$. We know that the discrete energy inequality (4.25) holds, that is,

$$\begin{aligned}\mathcal{E}_k(t) + \mathcal{V}_{\mathcal{H}}(H^p_k;0,t) \le \mathcal{E}(0) + \int_0^{t^j_k}\int_\Omega \mathbb{C}H^e_{k,sym}(\tau) : E\dot{w}(\tau)\;dxd\tau \\ - \int_0^{t^j_k}\langle\dot{\mathcal{L}}(\tau), u_k(\tau)\rangle\;d\tau - \int_0^{t^j_k}\langle\mathcal{L}_k(\tau), \dot{w}(\tau)\rangle\;d\tau + \varepsilon_k\,,\end{aligned}$$

where j is such that $t \in [t^j_k, t^{j+1}_k)$ and $\varepsilon_k \to 0$ as $k \to \infty$. Since $\|u_k(t)\|_{W^{1,\frac{3}{2}}}$ is uniformly bounded with respect to k, we have that

$$\begin{aligned}&|\langle\mathcal{L}_k(t), u_k(t)\rangle - \langle\mathcal{L}(t), u(t)\rangle| \le |\langle\mathcal{L}_k(t) - \mathcal{L}(t), u_k(t)\rangle| + |\langle\mathcal{L}(t), u_k(t) - u(t)\rangle| \\ \le&\|\mathcal{L}_k(t) - \mathcal{L}(t)\|_{W^{-1,3}}\|u_k(t)\|_{W^{1,\frac{3}{2}}} + |\langle\mathcal{L}(t), u_k(t) - u(t)\rangle| \to 0\,.\end{aligned} \tag{4.61}$$

Since Ψ_1 and Ψ_2 are weakly lower semicontinuous by convexity, (4.61) implies that

$$\mathcal{E}(t) \le \liminf_{k\to\infty} \mathcal{E}_k(t)\,.$$

In view of Proposition 3.2.4,

$$\mathcal{V}_{\mathcal{H}}(H^p;0,t) \le \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}}(H^p_k;0,t)\,.$$

Finally, the right-hand side of the discrete energy inequality converges, by dominated convergence, to

$$\mathcal{E}(0) + \int_0^t\int_\Omega \mathbb{C}H^e_{sym}(\tau) : E\dot{w}(\tau)\;dxd\tau - \int_0^t\langle\dot{\mathcal{L}}(\tau), u(\tau)\rangle\;d\tau - \int_0^t\langle\mathcal{L}(\tau), \dot{w}(\tau)\rangle\;d\tau\,.$$

In other words, passing to the limit we have proved that

$$\begin{aligned}\mathcal{E}(t) + \mathcal{V}_{\mathcal{H}}(H^p;0,t) \le \mathcal{E}(0) + \int_0^t\int_\Omega \mathbb{C}H^e_{sym}(\tau) : E\dot{w}(\tau)\;dxd\tau \\ - \int_0^t\langle\dot{\mathcal{L}}(\tau), u(\tau)\rangle\;d\tau - \int_0^t\langle\mathcal{L}(\tau), \dot{w}(\tau)\rangle\;d\tau\,.\end{aligned} \tag{4.62}$$

We now prove the converse inequality. Let us fix $t \in [0,T]$ and $h \in \mathbb{N}$. For

$j = 0, \dots, h$ let $s_h^j = \dfrac{j}{h} t$. Let us fix $j \in \{0, \dots, h-1\}$. The triplet

$$(u(s_h^{j+1}) - w(s_h^{j+1}) + w(s_h^j), H^e(s_h^{j+1}) - \nabla w(s_h^{j+1}) + \nabla w(s_h^j), H^p(s_h^{j+1}))$$

is admissible for the boundary value $w(s_h^j)$. Therefore, by the global stability condition (3.56), we obtain

$$\begin{aligned} &\Psi_1(H^e_{sym}(s_h^{j+1}) - Ew(s_h^{j+1}) + Ew(s_h^j)) + \Psi_2(\mathrm{curl}(H^p(s_h^{j+1}))) \\ &- \langle \mathcal{L}(s_h^j), u(s_h^{j+1}) - w(s_h^{j+1}) + w(s_h^j) \rangle + \mathcal{H}(H^p(s_h^{j+1}) - H^p(s_h^j)) \\ \geq &\Psi_1(H^e_{sym}(s_h^j)) + \Psi_2(\mathrm{curl}(H^p(s_h^j))) - \langle \mathcal{L}(s_h^j), u(s_h^j) \rangle \, . \end{aligned} \tag{4.63}$$

Equation (4.63) can be rewritten in the following form:

$$\begin{aligned} &\Psi_1(H^e_{sym}(s_h^{j+1})) + \Psi_2(\mathrm{curl}(H^p(s_h^{j+1}))) - \langle \mathcal{L}(s_h^{j+1}), u(s_h^{j+1}) \rangle \\ &+ \mathcal{H}(H^p(s_h^{j+1}) - H^p(s_h^j)) \geq \Psi_1(H^e_{sym}(s_h^j)) + \Psi_2(\mathrm{curl}(H^p(s_h^j))) \\ &- \langle \mathcal{L}(s_h^j), u(s_h^j) \rangle + \int_{s_h^j}^{s_h^{j+1}} \int_\Omega \mathbb{C} H^e_{sym}(s_h^{j+1}) : E\dot{w}(\tau) \; d\tau dx \\ &- \int_{s_h^j}^{s_h^{j+1}} \langle \dot{\mathcal{L}}(\tau), u(s_h^{j+1}) \rangle \; d\tau - \int_{s_h^j}^{s_h^{j+1}} \langle \mathcal{L}(s_h^{j+1}), \dot{w}(\tau) \rangle \; d\tau + \delta_{k,j} \, , \end{aligned} \tag{4.64}$$

where

$$\delta_{h,j} := -\Psi_1(Ew(s_h^{j+1}) - Ew(s_h^j)) + \langle \mathcal{L}(s_h^{j+1}) - \mathcal{L}(s_h^j), w(s_h^{j+1}) - w(s_h^j) \rangle \, .$$

Let us define $\overline{u}_h, \overline{H}^e_h$ and $\overline{\mathcal{L}}_h$ as the left-continuous piecewise constant interpolations of u, H^e and $\mathcal{L}$, respectively, obtained by taking the value at the point s_h^{j+1} on each interval $(s_h^j, s_h^{j+1}]$. Then, summing up (4.64) for $j = 0, \dots, h-1$ we obtain

$$\begin{aligned} \mathcal{E}(t) + \sum_{j=0}^{h-1} \mathcal{H}(H^p(s_h^{j+1}) - H^p(s_h^j)) \geq \mathcal{E}(0) + \int_0^t \int_\Omega \mathbb{C} \overline{H}^e_{h,sym}(\tau) : E\dot{w}(\tau) \; d\tau dx \\ - \int_0^t \langle \dot{\mathcal{L}}(\tau), \overline{u}_h(\tau) \rangle \; d\tau - \int_0^t \langle \overline{\mathcal{L}}_h(\tau), \dot{w}(\tau) \rangle \; d\tau + \sum_{j=0}^{h-1} \delta_{h,j} \, . \end{aligned}$$

By definition of $\mathcal{V}_\mathcal{H}$ it follows immediately that

$$\begin{aligned} &\mathcal{E}(t) + \mathcal{V}_\mathcal{H}(H^p; 0, t) \geq \mathcal{E}(0) + \int_0^t \int_\Omega \mathbb{C} \overline{H}^e_{h,sym}(\tau) : E\dot{w}(\tau) \; d\tau dx \\ &- \int_0^t \langle \dot{\mathcal{L}}(\tau), \overline{u}_h(\tau) \rangle \; d\tau - \int_0^t \langle \overline{\mathcal{L}}_h(\tau), \dot{w}(\tau) \rangle \; d\tau + \sum_{j=0}^{h-1} \delta_{h,j} \, . \end{aligned} \tag{4.65}$$

We would like to pass to the limit in (4.65). Note that, arguing as in the proof of Proposition 4.2.2, we can show that $\sum\limits_{j=0}^{h-1} \delta_{h,j} \to 0$, as $h \to \infty$. By the absolute continuity of $\mathcal{L}$ we already know that

$$\overline{\mathcal{L}}_h(s) \to \mathcal{L}(s) \quad \text{in } W^{-1,3}(\Omega;\mathbb{R}^3)\ \forall\ s \in [0,T]\,.$$

Hence, we can pass to the limit in (4.65) by dominated convergence if we prove that for almost every $s \in [0,T]$, the following convergences hold:

$$\overline{u}_h(s) \rightharpoonup u(s) \qquad \text{in } W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3)\,, \tag{4.66}$$

$$\overline{H}^e_{h,sym}(s) \rightharpoonup H^e_{sym}(s) \quad \text{in } L^2(\Omega;M^{3\times 3}_{sym})\,. \tag{4.67}$$

H^p_{sym} and H^p_{skew} are by construction functions of bounded variation with values in $BV(\Omega;M^{3\times 3}_{sym,D})$ and $L^1(\Omega;M^{3\times 3}_{skew})$, respectively. Therefore, they are almost everywhere continuous. Let s be a point of continuity for both maps. By Proposition 3.2.7 s is a point of continuity also for the map

$$H^e_{sym} : [0,T] \to L^2(\Omega;M^{3\times 3}_{sym})\,.$$

Finally, applying Theorem 2.2.2 and Theorem 2.2.3 s is a point of continuity for the map

$$u : [0,T] \to W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3)\,.$$

Therefore, convergences (4.66) and (4.67) hold almost everywhere. ∎

Chapter 5

Constitutive equations and the flow rule

In this chapter we will derive the constitutive equations for an energetic solution to the Gurtin model. The chapter is organized as follows: in the first section we will deduce the constitutive equations for the macro and micro stresses given by (3.6), (3.7), (3.9), and (3.10); in the last section we will show how to give a meaning to the flow rule (3.21) and (3.22). As a corollary we will prove that the evolution of the elastic strain and the Burgers vector are uniquely determined by the initial datum.

We will suppose for the whole chapter that the map

$$t \mapsto (u(t), H^e(t), H^p(t))$$

is an energetic solution for the Gurtin model. We define the stresses

$$T(t) := \mathbb{C}H^e(t)$$

and

$$R(t) := \mu L^2 \operatorname{curl}(H^p(t)), \tag{5.1}$$

as in (3.15) and (3.18).

5.1 Constitutive equations

Proposition 5.1.1. *For every $t \in [0,T]$ the stress $T(t) \in L^2_{\mathrm{div}}(\Omega, M^{3\times 3}_{sym})$ satisfies the following conditions:*

$$\begin{cases} -\operatorname{div}(T(t)) = f(t) & \text{in } L^2(\Omega;\mathbb{R}^3)\,, \\ \gamma_\nu(T(t)) = g(t) & \text{on } \Gamma_N\,, \end{cases}$$

where γ_ν is the normal trace. The second equation has to be intended in the following sense: for every $\psi \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$ such that $\psi = 0$ $\mathscr{H}^2$-a.e on Γ_D

$$\langle \gamma_\nu(T(t)), \psi\rangle = \int_{\Gamma_N} g(t)\cdot\psi \; d\mathscr{H}^2\,.$$

Proof. Let $t \in [0,T]$. Let $\phi \in H^1(\Omega;\mathbb{R}^3)$ such that $\psi = 0$ $\mathscr{H}^2$-a.e on Γ_D. Then

$$(u(t) + \varepsilon\phi, H^e + \varepsilon\nabla\phi, H^p) \in \mathcal{A}(w(t))\,,$$

hence by the global stability (3.56)

$$\begin{aligned}&\Psi_1(H^e_{sym}(t)) + \Psi_2(\mathrm{curl}(H^p(t))) - \langle \mathcal{L}(t), u(t)\rangle \\ &\le \Psi_1(H^e_{sym}(t) + \varepsilon E\phi) + \Psi_2(\mathrm{curl}(H^p(t))) - \langle \mathcal{L}(t), u(t) + \varepsilon\phi\rangle\,. \end{aligned} \tag{5.2}$$

Equation (5.2) can be rewritten in the following form:

$$\varepsilon^2\Psi_1(E\phi) + \varepsilon\int_\Omega T(t) : \nabla\phi\; dx - \varepsilon\langle \mathcal{L}(t), \phi\rangle \ge 0\,. \tag{5.3}$$

Dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0^+$ in (5.3) we deduce

$$\int_\Omega T(t) : \nabla\phi\; dx - \langle \mathcal{L}(t), \phi\rangle \ge 0\,.$$

If $\varepsilon \to 0^-$, we obtain

$$\int_\Omega T(t) : \nabla\phi\; dx - \langle \mathcal{L}(t), \phi\rangle \le 0\,.$$

Therefore, we conclude that for every $\phi \in H^1(\Omega;\mathbb{R}^3)$ such that $\phi = 0$ $\mathscr{H}^2$-a.e on Γ_D

$$\int_\Omega T(t) : \nabla\phi\; dx = \langle \mathcal{L}(t), \phi\rangle\,. \tag{5.4}$$

In particular, if $\phi \in \mathscr{D}(\Omega;\mathbb{R}^3)$, we have

$$\langle \mathrm{div}(T(t)), \phi\rangle = -\int_\Omega f(t)\cdot\phi\; dx\,.$$

Hence, $T(t) \in L^2_{\mathrm{div}}(\Omega; M^{3\times 3}_{sym})$ and

$$-\,\mathrm{div}(T(t)) = f(t) \quad \text{in } L^2(\Omega;\mathbb{R}^3)\,. \tag{5.5}$$

Since $T(t) \in L^2_{div}(\Omega; M^{3\times3}_{sym})$ the normal trace $\gamma_\nu(T(t))$ is well defined as an element of $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$. By (5.4) and (5.5), for every $\psi \in H^1(\Omega;\mathbb{R}^3)$ such that $\psi = 0$ $\mathscr{H}^2$-a.e on Γ_D, we have

$$\langle \gamma_\nu(T(t)), \psi\rangle = -\int_\Omega f(t)\cdot\psi\; dx + \int_\Omega T(t) : \nabla\psi\; dx = \int_{\Gamma_N} g(t)\cdot\psi\; d\mathscr{H}^2\,.$$

This concludes the proof. ∎

In order to prove the constitutive equations (3.9) and (3.10) we need to define the plastic stresses for an energetic solution to the Gurtin model. This is the aim of the next proposition.

Proposition 5.1.2. *There exist maps*

$$T^p : [0,T] \to L^\infty(\Omega; M_D^{3\times 3}),$$
$$K_{diss} : [0,T] \to L^\infty(\Omega; M_{D,sym}^{3\times 3\times 3}),$$
$$S : [0,T] \to \mathscr{M}_b(\Omega; M_{D,sym}^{3\times 3\times 3})^*,$$

such that for every $t \in [0,T]$ *and* $(v,e,p) \in \mathcal{A}(0)$

$$\begin{aligned}&\int_\Omega T(t) : e_{sym}\, dx + \int_\Omega R(t) : \mathrm{curl}(p)\, dx - \langle \mathcal{L}(t), v\rangle \\ &= -\int_\Omega T^p(t) : p\, dx - \int_\Omega K_{diss}(t) : \nabla^a p_{sym}\, dx - \langle S(t), \nabla^s p_{sym}\rangle. \end{aligned} \tag{5.6}$$

Moreover, for every $(A,B,C) \in L^1(\Omega; M_D^{3\times 3}) \times L^1(\Omega; M_{D,sym}^{3\times 3\times 3}) \times \mathscr{M}_b(\Omega; M_{D,sym}^{3\times 3\times 3})$

$$\begin{aligned}&\left|\int_\Omega T^p(t) : A\, dx + \int_\Omega K_{diss}(t) : B\, dx + \langle S(t), C\rangle\right| \\ &\leq Y_0 \|(A,B)\|_{\mathcal{H}} + hY_0|C|(\Omega) \quad \forall\ t \in [0,T]. \end{aligned} \tag{5.7}$$

Proof. By Lemma 3.2.1 we have that for every $(v,e,p) \in \mathcal{A}(0)$

$$\left|\int_\Omega T(t) : e_{sym}\, dx + \int_\Omega R(t) : \mathrm{curl}(p)\, dx - \langle \mathcal{L}(t), v\rangle\right| \leq \mathcal{H}(p). \tag{5.8}$$

In particular, the linear functional $\Phi : \mathcal{A}(0) \to \mathbb{R}$ given by

$$(v,e,p) \mapsto \int_\Omega T(t) : e_{sym}\, dx + \int_\Omega R(t) : \mathrm{curl}(p)\, dx - \langle \mathcal{L}(t), v\rangle$$

depends only on p. Let us denote by X the linear subspace of

$$L^1(\Omega; M_D^{3\times 3}) \times L^1(\Omega; M^{3\times 3\times 3}) \times \mathscr{M}_b(\Omega; M^{3\times 3\times 3}) \tag{5.9}$$

made of all triplets $(p, \nabla^a p_{sym}, \nabla^s p_{sym})$ such that $(v,e,p) \in \mathcal{A}(0)$ for some e, v. The functional Φ is well defined on X since it depends only on p and by (5.8) is continuous with respect to the strong topology of (5.9). Therefore, by the Hahn-Banach Theorem it admits a continuous extension defined on the space (5.9). Let us denote by $\tilde{\Phi}$ such extension. By the Hahn-Banach Theorem, (5.8) still holds on the whole space, that is,

$$\begin{aligned}&|\tilde{\Phi}(A,B,C)| \leq Y_0\|(A,B)\|_{\mathcal{H}} + hY_0|C|(\Omega) \\ &\forall\ (A,B,C) \in L^1(\Omega; M_D^{3\times 3}) \times L^1(\Omega; M^{3\times 3\times 3}) \times \mathscr{M}_b(\Omega; M^{3\times 3\times 3}).\end{aligned}$$

By the Riesz representation Theorem $\tilde{\Phi}$ can be represented by means of operators $T^p(t), K_{diss}(t), S(t)$ as in (5.7). Finally, (5.6) holds by construction. ■

Let K_{diss}, T^p, and S be the tensors given by Proposition 5.1.2 for the energetic solution

$$t \mapsto (u(t), H^e(t), H^p(t)).$$

Let us define the tensor $K(t) := K_{diss}(t) + K_{en}(t)$, where K_{en} is given by

$$(K_{en}(t))_{ijk} = \sum_h (R(t))_{hi}\epsilon_{hkj} - \frac{1}{3}\delta_{ij}\sum_{h,m}(R(t))_{hm}\epsilon_{hkm}\,. \tag{5.10}$$

Note that, by definition, $K(t) \in L^2(\Omega; M_D^{3\times 3})$ for every $t \in [0,T]$.

Lemma 5.1.1. *For every $t \in [0,T]$ and for every $A \in H^1(\Omega; M_D^{3\times 3})$ the following holds:*

$$R(t) : \mathrm{curl}(A) = K_{en}(t) : \nabla A\,. \tag{5.11}$$

Proof. Let $t \in [0,T]$. Let us denote by $P(t)$ the tensor given by

$$P(t) := \left(\sum_h (R(t))_{hi}\epsilon_{hkj}\right)_{ijk}$$

and let us call F the linear operator such that

$$(F(B))_{ij} = \sum_{p,q} \epsilon_{ipq} B_{jqp} \quad \forall\, B \in M^{3\times 3\times 3}\,.$$

It is immediate to see that

$$P(t) : B = R(t) : F(B) \quad \forall\, B \in M^{3\times 3\times 3}$$

Let $A \in H^1(\Omega; M_D^{3\times 3})$. Since $F(\nabla A) = \mathrm{curl}(A)$ by definition, we deduce

$$P(t) : \nabla A = R(t) : \mathrm{curl}(A)\,.$$

$K_{en}(t)$ is, by definition, the projection of $P(t)$ on the subspace of deviatoric tensors in the first two subscripts. Since A takes values in the deviatoric matrices, we obtain (5.11). ∎

Proposition 5.1.3. *For every $t \in [0,T]$ the plastic stress $K(t) \in L^2_{\mathrm{div}}(\Omega; M_D^{3\times 3})$. Moreover, $T^p(t)$ and $K(t)$ satisfy*

$$\begin{cases} T^p(t) = T_D(t) + \mathrm{div}(K(t)) & \text{in } L^2(\Omega; M_D^{3\times 3})\,, \\ \gamma_\nu(K(t)) = 0 & \text{in } H^{-\frac{1}{2}}(\partial\Omega; M_D^{3\times 3})\,. \end{cases}$$

Proof. Let $\psi \in C^\infty(\overline{\Omega}; M_D^{3\times 3})$ and $t \in [0,T]$. Clearly $(0, -\psi, \psi) \in \mathcal{A}(0)$, hence by Proposition 5.1.2 we have

$$-\int_\Omega T(t) : \psi\, dx + \int_\Omega R(t) : \mathrm{curl}(\psi)\, dx + \int_\Omega T^p(t) : \psi\, dx + \int_\Omega K_{diss}(t) : \nabla\psi\, dx = 0\,. \tag{5.12}$$

By Lemma 5.1.1 we can rewrite (5.12) in the following form:

$$-\int_\Omega T_D(t) : \psi\, dx + \int_\Omega T^p(t) : \psi\, dx + \int_\Omega K(t) : \nabla\psi\, dx = 0\,. \tag{5.13}$$

The previous equation implies that $K(t) \in L^2_{\rm div}(\Omega; M^{3\times3}_D)$ and

$$T^p(t) = T_D(t) + \mathrm{div}(K(t)) \quad \text{in } L^2(\Omega; M^{3\times3}_D)\,.$$

In particular the normal trace $\gamma_\nu(K(t))$ is well defined in $H^{-\frac{1}{2}}(\partial\Omega; M^{3\times3}_D)$ and is equal to 0 by (5.13). This concludes the proof. ∎

5.2 The flow rule

In this section we will show that the flow rule given in (3.21) and (3.22) is satisfied. To establish this result we will need to prove the existence of the time derivatives for an energetic solution to the Gurtin model. Indeed, we will see that all the involved maps are absolutely continuous. We start by proving the constraint (3.24).

Proposition 5.2.1. *The plastic stresses T^p and K_{diss} satisfy the following constraint: for every $t \in [0,T]$*

$$\sqrt{|T^p_{sym}(t)|^2 + \frac{1}{\chi}|T^p_{skew}(t)|^2 + \frac{1}{h^2}|K_{diss}(t)|^2} \le Y_0 \quad \textit{a.e. in } \Omega\,.$$

Moreover, $\|S(t)\|_{\mathscr{M}_b^*} \le hY_0$ for every $t \in [0,T]$.

Proof. Let $t \in [0,T]$. Choosing $A = 0$, $B = 0$ in (5.7), we deduce

$$|\langle S(t), C\rangle| \le hY_0 \|C\|_{\mathscr{M}_b} \quad \forall\, C \in \mathscr{M}_b(\Omega; M^{3\times3\times3}_{D,sym})\,.$$

Hence,

$$\|S(t)\|_{\mathscr{M}_b^*} \le hY_0\,.$$

Similarly, setting $C = 0$ in (5.7), we have

$$\left| \int_\Omega T^p(t) : A \; dx + \int_\Omega K_{diss}(t) : B \; dx \right| \le Y_0 \|(A,B)\|_{\mathcal{H}}$$

for every $(A,B) \in L^1(\Omega; M^{3\times3}_D) \times L^1(\Omega; M^{3\times3\times3}_{D,sym})$. Recalling (4.12) we deduce that

$$\|(T^p(t), K_{diss}(t))\|^*_{\mathcal{H}} \le Y_0\,.$$

∎

We now show the absolute continuity of the solution.

Theorem 5.2.1. *The maps*

$$\begin{aligned}
&u : [0,T] \to W^{1,\frac{3}{2}}(\Omega; \mathbb{R}^3)\,,\\
&H^e_{sym} : [0,T] \to L^2(\Omega; M^{3\times3}_{sym})\,,\\
&H^p_{sym} : [0,T] \to \mathrm{BV}(\Omega; M^{3\times3}_{D,sym})\,,\\
&H^p_{skew} : [0,T] \to L^1(\Omega; M^{3\times3}_{skew})\,,
\end{aligned}$$

$$\operatorname{curl}(H^p) : [0,T] \to L^2(\Omega; M^{3\times 3})$$

are absolutely continuous.

Proof. We fix $t_1, t_2 \in [0,T]$ where $t_1 < t_2$. By the energy balance (3.57) we have that

$$\begin{aligned}
&\Psi_1(H^e_{sym}(t_2)) - \Psi_1(H^e_{sym}(t_1)) + \Psi_2(\operatorname{curl}(H^p(t_2))) - \Psi_2(\operatorname{curl}(H^p(t_1)))\\
&- \langle \mathcal{L}(t_2), u(t_2)\rangle + \langle \mathcal{L}(t_1), u(t_1)\rangle + \mathcal{H}(H^p(t_2) - H^p(t_1)) \qquad (5.14)\\
&\le \int_{t_1}^{t_2}\int_\Omega \mathbb{C}H^e_{sym}(\tau) : E\dot{w}(\tau)\; dxd\tau - \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\; d\tau - \int_{t_1}^{t_2}\langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\; d\tau\,.
\end{aligned}$$

The triplet (v, e, p) defined by

$$\begin{aligned}
v &= u(t_2) - u(t_1) - (w(t_2) - w(t_1))\,,\\
e &= H^e(t_2) - H^e(t_1) - (\nabla w(t_2) - \nabla w(t_1))\,,\\
p &= H^p(t_2) - H^p(t_1)\,,
\end{aligned}$$

is admissible for the zero boundary value; hence, by Lemma 3.2.1, we have

$$\begin{aligned}
&- \int_\Omega \mathbb{C}H^e_{sym}(t_1) : (H^e_{sym}(t_2) - H^e_{sym}(t_1) - (Ew(t_2) - Ew(t_1)))\; dx\\
&- \mu L^2 \int_\Omega \operatorname{curl}(H^p(t_1)) : (\operatorname{curl}(H^p(t_2) - \operatorname{curl}(H^p(t_1)))\; dx\\
&+ \langle \mathcal{L}(t_1), u(t_2) - u(t_1) - (w(t_2) - w(t_1))\rangle \le \mathcal{H}(H^p(t_2) - H^p(t_1))\,. \qquad (5.15)
\end{aligned}$$

Combining (5.14) and (5.15), we obtain

$$\begin{aligned}
&\Psi_1(H^e_{sym}(t_2) - H^e_{sym}(t_1)) + \Psi_2(\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1)))\\
&\le \int_{t_1}^{t_2}\int_\Omega \mathbb{C}H^e_{sym}(\tau) : E\dot{w}(\tau)\; dxd\tau - \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\; d\tau - \int_{t_1}^{t_2}\langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\; d\tau\\
&+ \langle \mathcal{L}(t_2) - \mathcal{L}(t_1), u(t_2)\rangle + \langle \mathcal{L}(t_1), w(t_2) - w(t_1)\rangle\\
&- \int_\Omega \mathbb{C}H^e_{sym}(t_1) : (Ew(t_2) - Ew(t_1))\; dx\\
&= \int_{t_1}^{t_2}\int_\Omega (\mathbb{C}H^e_{sym}(\tau) - \mathbb{C}H^e_{sym}(t_1)) : E\dot{w}(\tau)\; dxd\tau\\
&- \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}(\tau), u(\tau) - u(t_2)\rangle\; d\tau - \int_{t_1}^{t_2}\langle \mathcal{L}(\tau) - \mathcal{L}(t_1), \dot{w}(\tau)\rangle\; d\tau\,. \qquad (5.16)
\end{aligned}$$

By Proposition 3.2.6 we can rewrite (5.16) as follows:

$$\begin{aligned}
&\Psi_1(H^e_{sym}(t_2) - H^e_{sym}(t_1)) + \Psi_2(\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1)))\\
&\le \int_{t_1}^{t_2}\int_\Omega (\mathbb{C}H^e_{sym}(\tau) - \mathbb{C}H^e_{sym}(t_1)) : E\dot{w}(\tau)\; dxd\tau\\
&- \int_{t_1}^{t_2}\int_\Omega \dot{\rho}(\tau) : (H^e_{sym}(\tau) - H^e_{sym}(t_2))\; dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&- \int_{t_1}^{t_2} \int_\Omega \dot\rho_D(\tau) : (H^p_{sym}(\tau) - H^p_{sym}(t_2)) \; dx d\tau \\
&- \int_{t_1}^{t_2} \int_\Omega (\rho(\tau) - \rho(t_1)) : E\dot w(\tau) \; dx d\tau \, .
\end{aligned} \tag{5.17}
$$

Indeed, the previous inequality follows by observing that

$$
\begin{aligned}
&- \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}(\tau), w(\tau) \rangle \; d\tau + \int_{t_1}^{t_2} \int_\Omega \dot\rho(\tau) : Ew(\tau) \; dx d\tau + \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}(\tau), w(t_2) \rangle \; d\tau \\
&- \int_{t_1}^{t_2} \int_\Omega \dot\rho(\tau) : Ew(t_2) \; dx d\tau - \int_{t_1}^{t_2} \langle \mathcal{L}(\tau), \dot w(\tau) \rangle \; d\tau + \int_{t_1}^{t_2} \int_\Omega \rho(\tau) : E\dot w(\tau) \; dx d\tau \\
&+ \int_{t_1}^{t_2} \langle \mathcal{L}(t_1), \dot w(\tau) \rangle \; d\tau - \int_{t_1}^{t_2} \int_\Omega \rho(t_1) : E\dot w(\tau) \; dx d\tau \\
&= - \int_{t_1}^{t_2} \frac{d}{d\tau} [\langle \mathcal{L}(\tau), w(\tau) \rangle] \; d\tau + \int_\Omega \int_{t_1}^{t_2} \frac{d}{d\tau} [\rho(\tau) : Ew(\tau)] \; d\tau \\
&+ \int_{t_1}^{t_2} \frac{d}{d\tau} [\langle \mathcal{L}(\tau), w(t_2) \rangle] \; d\tau - \int_\Omega \int_{t_1}^{t_2} \frac{d}{d\tau} [\rho(\tau) : Ew(t_2)] d\tau \\
&+ \int_{t_1}^{t_2} \frac{d}{d\tau} [\langle \mathcal{L}(t_1), w(\tau) \rangle] \; d\tau - \int_\Omega \int_{t_1}^{t_2} \frac{d}{d\tau} [\rho(t_1) : Ew(\tau)] \; d\tau = 0 \, .
\end{aligned}
$$

By the Hölder inequality, (3.44), and (5.17) we deduce that

$$
\begin{aligned}
&\frac{\alpha_C}{2} \| H^e_{sym}(t_2) - H^e_{sym}(t_1)) \|^2_{L^2} + \frac{\mu L^2}{2} \| \operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1)) \|^2_{L^2} \\
&\le \beta_C \int_{t_1}^{t_2} \| H^e_{sym}(\tau) - H^e_{sym}(t_1) \|_{L^2} \| E\dot w(\tau) \|_{L^2} \; d\tau \\
&+ \int_{t_1}^{t_2} \| \dot\rho(\tau) \|_{L^2} \| H^e_{sym}(\tau) - H^e_{sym}(t_2) \|_{L^2} \; d\tau \\
&+ \int_{t_1}^{t_2} \| \dot\rho_D(\tau) \|_{L^\infty} \| H^p_{sym}(\tau) - H^p_{sym}(t_2) \|_{L^1} \; d\tau \\
&+ \int_{t_1}^{t_2} \| \rho(\tau) - \rho(t_1) \|_{L^2} \| E\dot w(\tau) \|_{L^2} \; d\tau \, .
\end{aligned} \tag{5.18}
$$

Let

$$
\phi(\tau) := \| \dot\rho(\tau) \|_{L^2} + \| \dot\rho_D(\tau) \|_{L^\infty} + \| E\dot w(\tau) \|_{L^2} \, .
$$

Given the regularity of ρ, ρ_D, and w we know that $\phi \in L^1(0,T)$. Note that

$$
\| \rho(\tau) - \rho(t_1) \|_{L^2} = \left\| \int_{t_1}^{\tau} \dot\rho(s) \; ds \right\|_{L^2} \le \int_{t_1}^{t_2} \| \dot\rho(s) \|_{L^2} \; ds \le \int_{t_1}^{t_2} \phi(s) \; ds \, . \tag{5.19}
$$

Therefore, by (5.18) and (5.19), there exists $C > 0$, depending only on $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, such that

$$
\begin{aligned}
&\| H^e_{sym}(t_2) - H^e_{sym}(t_1)) \|^2_{L^2} + \frac{\mu L^2}{2} \| \operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1)) \|^2_{L^2} \\
&\le C \int_{t_1}^{t_2} \phi(\tau) \| H^e_{sym}(\tau) - H^e_{sym}(t_1) \|_{L^2} \; d\tau
\end{aligned}
$$

$$+ C \int_{t_1}^{t_2} \phi(\tau) \|H^e_{sym}(\tau) - H^e_{sym}(t_2)\|_{L^2} \, d\tau$$

$$+ C \int_{t_1}^{t_2} \phi(\tau) \|H^p_{sym}(\tau) - H^p_{sym}(t_2)\|_{L^1} \, d\tau + C \left(\int_{t_1}^{t_2} \phi(\tau) \, d\tau \right)^2 . \tag{5.20}$$

Therefore, by the Cauchy inequality there exists a new constant $C > 0$, depending only on $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$, such that

$$\begin{aligned}
&\|H^e_{sym}(t_2) - H^e_{sym}(t_1))\|^2_{L^2} + \frac{\mu L^2}{2} \| \operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1))\|^2_{L^2} \\
&\leq C \int_{t_1}^{t_2} \phi(\tau) \|H^e_{sym}(t_2) - H^e_{sym}(\tau)\|_{L^2} \, d\tau \\
&+ C \int_{t_1}^{t_2} \phi(\tau) \|H^p_{sym}(t_2) - H^p_{sym}(\tau)\|_{L^1} \, d\tau + C \left(\int_{t_1}^{t_2} \phi(\tau) \, d\tau \right)^2 .
\end{aligned} \tag{5.21}$$

Let $s \in [t_1, t_2)$. By Proposition 4.1.3 we have that

$$\begin{aligned}
&\alpha \|H^p_{sym}(t_2) - H^p_{sym}(s)\|_{\mathrm{BV}} + \beta \|H^p_{skew}(t_2) - H^p_{skew}(s)\|_{L^1} \\
&\leq \mathcal{H}(H^p(t_2) - H^p(s)) - \int_\Omega \rho_D(t_2) : (H^p_{sym}(t_2) - H^p_{sym}(s)) \, dx .
\end{aligned} \tag{5.22}$$

Putting $t_1 = s$ in (5.14), by (5.22) we deduce

$$\begin{aligned}
&\alpha \|H^p_{sym}(t_2) - H^p_{sym}(s)\|_{\mathrm{BV}} + \beta \|H^p_{skew}(t_2) - H^p_{skew}(s)\|_{L^1} \\
&\leq \Psi_1(H^e_{sym}(s)) - \Psi_1(H^e_{sym}(t_2)) + \Psi_2(\operatorname{curl}(H^p(s))) - \Psi_2(\operatorname{curl}(H^p(t_2))) \\
&+ \langle \mathcal{L}(t_2), u(t_2) \rangle - \langle \mathcal{L}(s), u(s) \rangle + \int_s^{t_2} \int_\Omega \mathbb{C} H^e_{sym}(\tau) : E\dot{w}(\tau) \, dx d\tau \\
&- \int_s^{t_2} \langle \dot{\mathcal{L}}(\tau), u(\tau) \rangle \, d\tau - \int_s^{t_2} \langle \mathcal{L}(\tau), \dot{w}(\tau) \rangle \, d\tau \\
&- \int_\Omega \rho_D(t_2) : (H^p_{sym}(t_2) - H^p_{sym}(s)) \, dx .
\end{aligned} \tag{5.23}$$

Applying Proposition 3.2.5 and Proposition 3.2.6 we infer from (5.23) that

$$\begin{aligned}
&\alpha \|H^p_{sym}(t_2) - H^p_{sym}(s)\|_{\mathrm{BV}} + \beta \|H^p_{skew}(t_2) - H^p_{skew}(s)\|_{L^1} \\
&\leq \Psi_1(H^e_{sym}(s)) - \Psi_1(H^e_{sym}(t_2)) + \Psi_2(\operatorname{curl}(H^p(s))) - \Psi_2(\operatorname{curl}(H^p(t_2))) \\
&+ \int_\Omega \rho(t_2) : (H^e_{sym}(t_2) - H^e_{sym}(s)) \, dx + \int_\Omega (\rho(t_2) - \rho(s)) : H^e_{sym}(s) \, dx \\
&+ \int_\Omega (\rho_D(t_2) - \rho_D(s)) : H^p_{sym}(s) \, dx - \int_s^{t_2} \int_\Omega \dot{\rho}(\tau) : H^e_{sym}(\tau) \, dx d\tau \\
&- \int_s^{t_2} \int_\Omega \dot{\rho}_D(\tau) : H^p_{sym}(\tau) dx d\tau + \int_s^{t_2} \int_\Omega (\mathbb{C} H^e_{sym}(\tau) - \rho(\tau)) : E\dot{w}(\tau) \, dx d\tau .
\end{aligned} \tag{5.24}$$

Indeed, we have the following identity:

$$\langle \mathcal{L}(t_2), w(t_2) \rangle - \int_\Omega \rho(t_2) : Ew(t_2) \, dx - \langle \mathcal{L}(s), w(s) \rangle + \int_\Omega \rho(s) : Ew(s) \, dx$$

$$
\begin{aligned}
&-\int_s^{t_2} \langle \dot{\mathcal{L}}(\tau), w(\tau)\rangle \; d\tau + \int_s^{t_2}\int_\Omega \dot\rho(\tau) : Ew(\tau)\; dxd\tau - \int_s^{t_2} \langle \mathcal{L}(\tau), \dot w(\tau)\rangle \; d\tau \\
&+ \int_s^{t_2}\int_\Omega \rho(\tau) : E\dot w(\tau)\; dxd\tau = \langle \mathcal{L}(t_2), w(t_2)\rangle - \int_\Omega \rho(t_2) : Ew(t_2)\; dx \\
&- \langle \mathcal{L}(s), w(s)\rangle + \int_\Omega \rho(s) : Ew(s)\; dx - \int_s^{t_2} \frac{d}{d\tau}[\langle \mathcal{L}(\tau), w(\tau)\rangle]\; d\tau \\
&+ \int_\Omega \int_s^{t_2} \frac{d}{d\tau}[\rho(\tau) : Ew(\tau)]\; dxd\tau = 0\,.
\end{aligned}
$$

By the Hölder inequality and (5.24) we obtain

$$
\begin{aligned}
&\alpha\|H^p_{sym}(t_2) - H^p_{sym}(s)\|_{\mathrm{BV}} + \beta\|H^p_{skew}(t_2) - H^p_{skew}(s)\|_{L^1} \\
&\le \frac{\beta_{\mathbb{C}}}{2}\|H^e_{sym}(t_2) - H^e_{sym}(s)\|_{L^2}\|H^e_{sym}(t_2) + H^e_{sym}(s)\|_{L^2} \\
&+ \frac{\mu L^2}{2}\|\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(s))\|_{L^2}\|\operatorname{curl}(H^p(t_2)) + \operatorname{curl}(H^p(s))\|_{L^2} \\
&+ \|\rho(t_2)\|_{L^2}\|H^e_{sym}(t_2) - H^e_{sym}(s)\|_{L^2} \\
&+ \|\rho(t_2) - \rho(s)\|_{L^2}\|H^e_{sym}(s)\|_{L^2} + \|\rho_D(t_2) - \rho_D(s)\|_{L^\infty}\|H^p_{sym}(s)\|_{L^1} \\
&+ \int_s^{t_2} \|\dot\rho(\tau)\|_{L^2}\|H^e_{sym}(\tau)\|_{L^2}\; d\tau + \int_s^{t_2} \|\dot\rho_D(\tau)\|_{L^\infty}\|H^p_{sym}(\tau)\|_{L^1}\; d\tau \\
&+ \int_s^{t_2} \|\mathbb{C}H^e_{sym}(\tau) - \rho(\tau)\|_{L^2}\|E\dot w(\tau)\|_{L^2}\; d\tau\,. \qquad (5.25)
\end{aligned}
$$

Note that

$$
\|\rho(t_2) - \rho(s)\|_{L^2} = \left\|\int_s^{t_2} \dot\rho(\tau)\; d\tau\right\|_{L^2} \le \int_s^{t_2} \|\dot\rho(\tau)\|_{L^2}\; d\tau\,, \qquad (5.26)
$$

$$
\|\rho_D(t_2) - \rho_D(s)\|_{L^\infty} \le \int_s^{t_2} \|\dot\rho_D(\tau)\|_{L^\infty}\; d\tau\,. \qquad (5.27)
$$

By the absolute continuity of ρ we also have

$$
\sup_{t\in[0,T]} \|\rho(t)\|_{L^2} < +\infty\,. \qquad (5.28)
$$

Moreover, by Proposition 4.3.1 and Remark 4.3.1 we have

$$
\mathcal{V}(H^p_{sym}; 0, T; L^1) \le 2\frac{C_2}{M}\,, \qquad (5.29)
$$

$$
\sup_{t\in[0,T]} \|H^e_{sym}(t)\|_{L^2} \le C_1 \qquad (5.30)
$$

$$
\sup_{t\in[0,T]} \sqrt{\frac{\mu}{2}} L\|\operatorname{curl}(H^p(t))\|_{L^2} \le \sqrt{C_2}\,, \qquad (5.31)
$$

where M is the constant in (3.50). Taking into account (5.26)-(5.31) equation (5.25) can be rewritten in the following form:

$$
\alpha\|H^p_{sym}(t_2) - H^p_{sym}(s)\|_{\mathrm{BV}} + \beta\|H^p_{skew}(t_2) - H^p_{skew}(s)\|_{L^1}
$$

$$\leq C'\|H^e_{sym}(t_2) - H^e_{sym}(s)\|_{L^2} + C'\sqrt{\frac{\mu}{2}}L\|\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(s))\|_{L^2}$$
$$+ C'\int_s^{t_2}\phi(\tau)\, d\tau \quad \forall\, s \in [t_1, t_2]\,, \tag{5.32}$$

where $C' > 0$ depends only on $\beta_{\mathbb{C}}, C_1, C_2, \rho, M$, and $\|H^p_{sym}(0)\|_{L^1}$. Note that, using Remark 4.1.1 in place of Proposition 4.1.3, the same argument provides us with an estimate for the L^1 norm of $H^p_{sym}(t_2) - H^p_{sym}(t_1)$ independent of the constant α (hence, independent of h). More precisely,

$$\|H^p_{sym}(t_2) - H^p_{sym}(s)\|_{L^1} \leq C''\|H^e_{sym}(t_2) - H^e_{sym}(s)\|_{L^2} \tag{5.33}$$
$$+ C''\sqrt{\frac{\mu}{2}}L\|\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(s))\|_{L^2} + C''\int_s^{t_2}\phi(\tau)\, d\tau \quad \forall\, s \in [t_1, t_2]\,,$$

where $C'' > 0$ depends only on $\beta_{\mathbb{C}}, C_1, C_2, \rho, M$, and $\|H^p_{sym}(0)\|_{L^1}$. Combining (5.21) and (5.33) we deduce

$$\|H^e_{sym}(t_2) - H^e_{sym}(t_1))\|^2_{L^2} + \frac{\mu L^2}{2}\|\operatorname{curl}(H^p(t_2)) - \operatorname{curl}(H^p(t_1))\|^2_{L^2}$$
$$\leq C\int_{t_1}^{t_2}\phi(\tau)\|H^e_{sym}(\tau) - H^e_{sym}(t_1)\|_{L^2}\, dx \tag{5.34}$$
$$+ C\sqrt{\frac{\mu}{2}}L\int_{t_1}^{t_2}\phi(\tau)\|\operatorname{curl}(H^p(\tau)) - \operatorname{curl}(H^p(t_1))\|_{L^2}\, d\tau + C\left(\int_{t_1}^{t_2}\phi(\tau)\, d\tau\right)^2,$$

where $C > 0$ depends on $\alpha_{\mathbb{C}}, \beta_{\mathbb{C}}, C_1, C_2, \rho, M$, and $\|H^p_{sym}(0)\|_{L^1}$. Applying Lemma 5.3 in [15] we infer from (5.34)

$$\|H^e_{sym}(\tau) - H^e_{sym}(t_1)\|_{L^2} + \sqrt{\frac{\mu}{2}}L\|\operatorname{curl}(H^p(\tau)) - \operatorname{curl}(H^p(t_1))\|_{L^2}$$
$$\leq C\int_{t_1}^{t_2}\phi(\tau)\, d\tau$$

for some $C > 0$ that depends only upon $\alpha_{\mathbb{C}}, \beta_{\mathbb{C}}, C_1, C_2, \rho, M$, and $\|H^p_{sym}(0)\|_{L^1}$. Hence H^e_{sym} and $\operatorname{curl}(H^p)$ are absolutely continuous. By (5.32) we deduce that H^p_{sym} and H^p_{skew} are absolutely continuous, too. Finally, applying Theorem 2.2.2 and Theorem 2.2.4 we conclude that u is absolutely continuous. ■

Since u, H^e_{sym}, and $\operatorname{curl}(H^p)$ are absolutely continuous with values in a reflexive space, the time derivatives $\dot{u}$, $\dot{H}^e_{sym}$, and $\operatorname{curl}(\dot{H}^p)$ exist almost everywhere. Similarly, if we consider H^p_{skew} as a map with values in $\mathscr{M}_b(\Omega; M^{3\times3}_{skew})$, the time derivatives $\dot{H}^p_{sym}$ and $\dot{H}^p_{skew}$ exist as the weak* limit of the difference quotients. In particular, we have proved the following corollary.

Corollary 5.2.1. *There exists a positive constant K_1, independent of χ, h, and L, such that for almost every $t \in [0, T]$*

$$\|\dot{u}(t)\|_{W^{1,\frac{3}{2}}} \leq (K_1 + \frac{1}{\alpha}K_1)\phi(t)\,,$$
$$\|\dot{H}^e_{sym}(t)\|_{L^2} \leq K_1\phi(t)\,,$$

$$\begin{aligned}
&\|\operatorname{curl}(\dot H^p(t))\|_{L^2} \le K_1\phi(t)\,,\\
&\|\dot H^p_{sym}(t)\|_{\mathrm{BV}} \le \frac{1}{\alpha}K_1\phi(t)\,,\\
&\|\dot H^p_{skew}(t)\|_{\mathscr{M}_b} \le \frac{1}{\beta}K_1\phi(t)\,,\\
&\|\dot H^p_{sym}(t)\|_{L^1} \le K_1\phi(t)\,,
\end{aligned}$$

where

$$\phi(t) := \|\dot\rho(t)\|_{L^2} + \|\dot\rho_D(t)\|_{L^\infty} + \|E\dot w(t)\|_{L^2}$$

and α, β are the constants defined in (4.14) and (4.15). Moreover, by Proposition 4.3.1 the constant K_1 can be chosen with the following form:

$$K_1 = K_1'|\psi(H^e_{sym}(0), \operatorname{curl}(H^p(0)))| + K_1'\|H^e_{sym}(0)\|_{L^2} + K_1'\|H^p_{sym}(0)\|_{L^1} + K_1'\,,$$

where K_1' does not depend on the initial triplet.

As a remark, note that for almost every $t \in [0,T]$ the triplet

$$(\dot u(t), \dot H^e(t), \dot H^p(t))$$

is admissible for the boundary value $\dot w(t)$.

Proposition 5.2.2. *Let $t \in [0,T]$ be such that the time derivatives $\dot H^e_{sym}(t)$, $\dot H^p_{sym}(t)$, $\dot H^p_{skew}(t)$, $\dot u(t)$, and $\dot w(t)$ exist. Then*

$$\begin{aligned}
\mathcal{H}(\dot H^p(t)) = &-\int_\Omega T(t) : (\dot H^e_{sym}(t) - E\dot w(t))\,dx\\
&-\int_\Omega R(t) : \operatorname{curl}(\dot H^p(t))\,dx + \langle \mathcal{L}(t), \dot u(t) - \dot w(t)\rangle
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}(\dot H^p(t)) = &\int_\Omega T^p(t) : \dot H^p\,dx + \int_\Omega K_{diss}(t) : \nabla^a \dot H^p_{sym}(t)\,dx\\
&+ \langle S(t), \nabla^s \dot H^p_{sym}(t)\rangle\,.
\end{aligned}$$

Proof. By Theorem 7.1 in [15] we have for every $s \in [0,T]$

$$\mathcal{V}_{\mathcal{H}}(H^p; 0, s) = \int_0^s \mathcal{H}(\dot H^p(\tau))\,d\tau\,.$$

Therefore, differentiating the energy balance (3.57) with respect to time at t we obtain

$$\begin{aligned}
&\int_\Omega T(t) : \dot H^e_{sym}(t)\,dx + \int_\Omega R(t) : \operatorname{curl}(\dot H^p(t))\,dx - \langle \dot{\mathcal{L}}(t), u(t)\rangle\\
&- \langle \mathcal{L}(t), \dot u(t)\rangle + \mathcal{H}(\dot H^p(t)) = \int_\Omega T(t) : E\dot w(t)\,dx - \langle \dot{\mathcal{L}}(t), u(t)\rangle - \langle \mathcal{L}(t), \dot w(t)\rangle\,.
\end{aligned}$$

This proves the first part of the proposition. The triplet

$$(\dot{u}(t)-\dot{w}(t), \dot{H}^e(t)-\nabla\dot{w}(t), \dot{H}^p(t))$$

is admissible for the zero boundary value, hence the second part of the thesis follows from (5.6). ∎

Corollary 5.2.2. *Let*

$$t \mapsto (u(t), H^e(t), H^p(t))$$

be an energetic solution to the Gurtin model. The maps

$$t \mapsto H^e_{sym}(t)\,, \quad t \mapsto \operatorname{curl}(H^p(t))$$

are uniquely determined by the initial triplet $(u(0), H^e(0), H^p(0))$.

Proof. Let

$$t \mapsto (u_1(t), H^e_1(t), H^p_1(t))\,, \quad t \mapsto (u_2(t), H^e_2(t), H^p_2(t))$$

be two energetic solutions to the Gurtin model with the same initial datum. Let $T_1(t) := \mathbb{C}H^e_{1,sym}(t)$ and $T_2(t) := \mathbb{C}H^e_{2,sym}(t)$. Let $T^p_1, K_{1,diss}$, and S_1 be the plastic stresses given in Proposition 5.1.2 for the solution

$$t \mapsto (u_1(t), H^e_1(t), H^p_1(t))\,.$$

Similarly, we define $T^p_2, K_{2,diss}$, and S_2. Finally, let $R_1(t) = \mu L^2 \operatorname{curl}(H^p_1(t))$ and $R_2(t) = \mu L^2 \operatorname{curl}(H^p_2(t))$. By Proposition 5.2.2 and Proposition 5.1.2 we deduce that for almost every $t \in [0,T]$

$$\begin{aligned}
&-\int_\Omega T_1(t) : (\dot{H}^e_{1,sym}(t) - E\dot{w}(t))\; dx - \int_\Omega R_1(t) : \operatorname{curl}(\dot{H}^p_1(t))\; dx \\
&+ \langle \mathcal{L}(t), \dot{u}_1(t) - \dot{w}(t)\rangle = \mathcal{H}(\dot{H}^p_1(t)) \geq \int_\Omega T^p_2(t) : \dot{H}^p_1(t)\; dx \\
&+ \int_\Omega K_{2,diss}(t) : \nabla^a \dot{H}^p_{1,sym}(t)\; dx + \langle S_2(t), \nabla^s \dot{H}^p_{1,sym}(t)\rangle \\
&= -\int_\Omega T_2(t) : (\dot{H}^e_{1,sym}(t) - E\dot{w}(t))\; dx - \int_\Omega R_2(t) : \operatorname{curl}(\dot{H}^p_1(t))\; dx \\
&+ \langle \mathcal{L}(t), \dot{u}_1(t) - \dot{w}(t)\rangle\,.
\end{aligned}$$

Therefore, for almost every $t \in [0,T]$

$$\begin{aligned}
&-\int_\Omega (T_2(t) - T_1(t)) : (\dot{H}^e_{1,sym}(t) - E\dot{w}(t))\; dx \\
&-\int_\Omega (R_2(t) - R_1(t)) : \operatorname{curl}(\dot{H}^p_1(t))\; dx \leq 0\,.
\end{aligned} \tag{5.35}$$

With the same argument one can prove that for almost every $t \in [0,T]$

$$-\int_\Omega (T_1(t) - T_2(t)) : (\dot{H}^e_{2,sym}(t) - E\dot{w}(t))\; dx$$

$$- \int_\Omega (R_1(t) - R_2(t)) : \operatorname{curl}(\dot H_2^p(t)) \, dx \le 0 \,. \tag{5.36}$$

Summing up (5.35) and (5.36) we have

$$\begin{aligned} &\int_\Omega (T_2(t) - T_1(t)) : (\dot H^e_{2,sym}(t) - \dot H^e_{1,sym}(t)) \, dx \\ &+ \int_\Omega (R_2(t) - R_1(t)) : (\operatorname{curl}(\dot H_2^p(t)) - \operatorname{curl}(\dot H_1^p(t))) \, dx \le 0 \quad \text{for a.e } t \in [0,T] \,, \end{aligned} \tag{5.37}$$

that is

$$\frac{d}{dt} \Big(\Psi_1(H^e_{2,sym}(t) - H^e_{1,sym}(t)) + \Psi_2(\operatorname{curl}(H_2^p(t)) - \operatorname{curl}(H_1^p(t))) \Big) \le 0 \,.$$

for almost every $t \in [0,T]$. Since at $t = 0$ we have

$$\Psi_1(H^e_{2,sym}(0) - H^e_{1,sym}(0)) + \Psi_2(\operatorname{curl}(H_2^p(0)) - \operatorname{curl}(H_1^p(0))) = 0$$

the proof is concluded. ∎

We are now in the position to derive the flow rule for an energetic solution to the Gurtin model.

Theorem 5.2.2 (Flow rule in a weak form)**.** *Let $t \in [0,T]$ be such that the time derivatives $\dot H^p_{sym}$ and $\dot H^p_{skew}$ exist. Then:*

1. *for every $(A,B) \in L^\infty(\Omega; M_D^{3\times3}) \times L^\infty(\Omega; M_D^{3\times3\times3})$ such that $\|(A,B)\|^*_{\mathcal H} \le Y_0$ there holds*

$$\int_\Omega (T^p(t) - A) : \dot H^p(t) \, dx + \int_\Omega (K_{diss}(t) - B) : \nabla^a \dot H^p_{sym}(t) \, dx \ge 0 \,;$$

2. *for every $C \in \mathscr{M}_b(\Omega; M^{3\times3\times3})^*$ such that $\|C\|_{\mathscr{M}_b^*} \le hY_0$ there holds*

$$\langle S(t) - C, \nabla^s \dot H^p_{sym}(t) \rangle \ge 0 \,.$$

Proof. Let

$$X := L^1(\Omega; M^{3\times3}) \times L^1(\Omega; M^{3\times3\times3}) \times \mathscr{M}_b(\Omega; M^{3\times3\times3}) \,.$$

We define the map

$$\Psi : X \to [0,\infty) : (A,B,C) \mapsto Y_0 \|(A,B)\|_{\mathcal H} + hY_0 |C|(\Omega)$$

and the set

$$\begin{aligned} \mathcal{K} := \Big\{ (A,B,C) \in L^\infty(\Omega; M^{3\times3}) \times L^\infty(\Omega; M^{3\times3\times3}) \times \mathscr{M}_b(\Omega; M^{3\times3\times3})^* \; : \\ \|(A,B)\|^*_{\mathcal H} \le Y_0 \text{ and } \|C\|_{\mathscr{M}_b^*} \le hY_0 \Big\} \,. \end{aligned}$$

By (4.11) we have

$$\Psi(D,E,F) \geq \int_\Omega A : D \; dx + \int_\Omega B : E \; dx + \langle C, F \rangle \tag{5.38}$$

for every $(A,B,C) \in \mathcal{K}$ and $(D,E,F) \in X$. Note that

$$(\dot H^p(t), \nabla^a \dot H^p_{sym}(t), \nabla^s \dot H^p_{sym}(t)) \in X\,.$$

Moreover,

$$\Psi(\dot H^p(t), \nabla^a \dot H^p_{sym}(t), \nabla^s \dot H^p_{sym}(t)) = \mathcal{H}(\dot H^p(t))\,.$$

Hence, by Proposition 5.2.2 and (5.38) we deduce

$$\int_\Omega (T^p(t) - A) : \dot H^p(t) \; dx + \int_\Omega (K_{diss}(t) - B) : \nabla^a \dot H^p_{sym}(t) \; dx \\ + \langle S - C, \nabla^s \dot H^p_{sym}(t) \rangle \geq 0 \tag{5.39}$$

for every $(A,B,C) \in \mathcal{K}$. By Proposition 5.2.1 we have

$$(T^p(t), K_{diss}(t), 0)\,,\ (0,0,S(t)) \in \mathcal{K}\,.$$

Hence, choosing

$$(A,B,C) = (T^p(t), K_{diss}(t), 0)\,, (A,B,C) = (0,0,S(t))$$

in (5.39) proves the thesis. ■

Theorem 5.2.3 (Flow rule)**.** *Let $t \in [0,T]$ be such that the derivatives $\dot H^p_{sym}$ and $\dot H^p_{skew}$ exists. Let $x \in \Omega$ be a Lebesgue point for $\dot H^p_{sym}(t)$, $\dot H^p_{skew}(t)$, $\nabla^a \dot H^p_{sym}(t)$, $T^p(t)$, and $K_{diss}(t)$. If*

$$\sqrt{|T^p_{sym}(t,x)|^2 + \frac{1}{\chi}|T^p_{skew}(t,x)|^2 + \frac{1}{h^2}|K_{diss}(t,x)|^2} < Y_0\,,$$

then $(\dot H^p(t,x), \nabla^a \dot H^p_{sym}(t,x)) = 0$. If instead

$$\sqrt{|T^p_{sym}(t,x)|^2 + \frac{1}{\chi}|T^p_{skew}(t,x)|^2 + \frac{1}{h^2}|K_{diss}(t,x)|^2} = Y_0\,,$$

then

$$\begin{cases} T^p(t,x) & = Y_0 \dfrac{\dot H^p_{sym}(t,x) + \chi \dot H^p_{skew}(t,x)}{\sqrt{|\dot H^p_{sym}(t,x)|^2 + \chi|\dot H^p_{skew}(t,x)|^2 + h^2|\nabla^a \dot H^p_{sym}(t,x)|^2}}\,, \\[2ex] K_{diss}(t,x) & = Y_0 \dfrac{h^2 \nabla^a \dot H^p_{sym}(t,x)}{\sqrt{|\dot H^p_{sym}(t,x)|^2 + \chi|\dot H^p_{skew}(t,x)|^2 + h^2|\nabla^a \dot H^p_{sym}(t,x)|^2}}\,. \end{cases}$$

Proof. Let $\mathcal{K}$ be the convex set

$$\mathcal{K} := \Big\{(A,B) \in M^{3\times 3}_D \times M^{3\times 3\times 3} \ : \ |(A,B)|^*_{\mathcal{H}} \leq Y_0\Big\}$$

For a given $0 < \varepsilon < 1$ we define

$$P_\varepsilon(A,B) := (T^p(t) + \varepsilon(A - T^p(t)), K_{diss}(t) + \varepsilon(B - K_{diss}(t))) \quad \forall\ (A,B) \in \mathcal{K}\,.$$

Let us fix a pair $(A,B) \in \mathcal{K}$. Note that we have

$$P_\varepsilon(A,B) \in L^\infty(\Omega; M^{3\times 3}_D) \times L^\infty(\Omega; M^{3\times 3\times 3})\,.$$

For a given $r > 0$ we define

$$F_r(A,B)(y) = \begin{cases} P_\varepsilon(A,B)(y) & \text{for a.e } y \in B_r(x)\,, \\ (T^p(t,y), K_{diss}(t,y)) & \text{for a.e } y \in \Omega \backslash B_r(x)\,. \end{cases}$$

By the convexity of $\mathcal{K}$, F_r is an admissible test function for the weak flow rule given in item 1 of Theorem 5.2.2. Therefore,

$$\frac{\varepsilon}{r^3}\Bigg[\int_{B_r(x)} (T^p(t) - A) : \dot{H}^p(t)\ dx + \int_{B_r(x)} (K_{diss}(t) - B) : \nabla^a \dot{H}^p_{sym}(t)\ dx\Bigg] \geq 0\,. \tag{5.40}$$

Passing to the limit as $r \to 0$ in (5.40) and dividing by ε we obtain

$$(T^p(t,x) - A) : \dot{H}^p(t,x) + (K_{diss}(t,x) - B) : \nabla^a \dot{H}^p(t,x) \geq 0 \quad \forall\ (A,B) \in \mathcal{K}\,.$$

Let us denote by $N_{\mathcal{K}}(A,B)$ the normal cone to $\mathcal{K}$ at the point (A,B). We have proved that

$$(\dot{H}^p(t,x), \nabla^a \dot{H}^p_{sym}(t,x)) \in N_{\mathcal{K}}(T^p(t,x), K^p_{diss}(t,x))\,.$$

If $(T^p(t,x), K^p_{diss}(t,x))$ lies in the interior of $\mathcal{K}$, then

$$(\dot{H}^p(t,x), \nabla^a \dot{H}^p_{sym}(t,x)) = 0\,.$$

If $(T^p(t,x), K^p_{diss}(t,x)) \in \partial\mathcal{K}$, then we have

$$\begin{cases} \dot{H}^p(t,x) & = \delta \dfrac{T^p_{sym}(t,x) + \dfrac{1}{\chi} T^p_{skew}(t,x)}{\sqrt{|T^p_{sym}(t,x)|^2 + \dfrac{1}{\chi}|T^p_{skew}(t,x)|^2 + \dfrac{1}{h^2}|K_{diss}(t,x)|^2}}\,, \\[2em] \nabla^a \dot{H}^p_{sym}(t,x) & = \delta \dfrac{\dfrac{1}{h^2} K_{diss}(t,x)}{\sqrt{|T^p_{sym}(t,x)|^2 + \dfrac{1}{\chi}|T^p_{skew}(t,x)|^2 + \dfrac{1}{h^2}|K_{diss}(t,x)|^2}} \end{cases}$$

for some $\delta > 0$, that is,

$$\begin{cases} T^p(t,x) & = Y_0 \dfrac{\dot{H}^p_{sym}(t,x) + \chi \dot{H}^p_{skew}(t,x)}{\delta}\,, \\[1.5em] K^p_{diss}(t,x) & = Y_0 \dfrac{h^2 \nabla^a \dot{H}^p_{sym}(t,x)}{\delta}\,. \end{cases}$$

To satisfy the constraint

$$\sqrt{|T^p_{sym}(t,x)|^2 + \frac{1}{\chi}|T^p_{skew}(t,x)|^2 + \frac{1}{h^2}|K_{diss}(t,x)|^2} = Y_0\,,$$

it must necessarily be that

$$\delta = \sqrt{|\dot{H}^p_{sym}(t,x)|^2 + \chi|\dot{H}^p_{skew}(t,x)|^2 + h^2|\nabla^a \dot{H}^p_{sym}(t,x)|^2}\,.$$

This concludes the proof. ∎

Chapter 6

Asymptotic analysis

In this chapter we will study the behavior of the energetic solutions to the Gurtin model as $\chi \to \infty$ and $L, h \to 0$.

In the first section we will show that, as $\chi \to \infty$, the solutions converge in a suitable sense to a solution to the model introduced by Gurtin and Anand in [11] and studied by Giacomini and Lussardi in [6].

In the second section we will prove that, as $h, L \to 0$, solutions to the Gurtin model converge in a suitable sense to a solution to the Prandtl-Reuss model of perfect plasticity studied in [15] by Dal Maso, DeSimone, and Mora. This result holds independently of the behavior χ.

6.1 Asymptotic behavior as $\chi \to \infty$

6.1.1 The Gurtin-Anand model

Giacomini and Lussardi studied in [6] the model proposed by Gurtin and Anand in [11] with the same energetic approach used in this manuscript. For the sake of brevity, we will refer to this model as GA. In contrast with the Gurtin model, in GA only the symmetric gradient is additively decomposed into an elastic and a plastic strain

$$Eu = H^e + H^p .$$

In particular, the GA model does not depend on the skew-symmetric part of H^p.

Given a boundary value $z \in H^1(\Omega; \mathbb{R}^3)$ we will say that a triplet (u, H^e, H^p) is admissible for the boundary value z in the GA model if and only if

$$u \in W^{1,\frac{3}{2}}(\Omega; \mathbb{R}^3) , \qquad H^e \in L^2(\Omega; M^{3\times 3}_{sym}) ,$$
$$H^p \in \mathrm{BV}(\Omega; M^{3\times 3}_{D,sym}) , \qquad \mathrm{curl}\, H^p \in L^2(\Omega; M^{3\times 3}) , \tag{6.1}$$

$$Eu = H^e + H^p , \tag{6.2}$$

$$u = z \text{ in } L^{\frac{3}{2}}(\Gamma_D; \mathbb{R}^3) . \tag{6.3}$$

We will denote by $\mathcal{A}_{GA}(z)$ the set of all admissible triplets for the boundary value z in the GA model.

We define the plastic dissipation functional for the GA model as follows:

$$\begin{aligned}\mathcal{H}_{GA} : &\mathrm{BV}(\Omega; M^{3\times 3}_{D,sym}) \to \mathbb{R} \\ &H^p \mapsto Y_0 \int_\Omega \sqrt{|H^p|^2 + h^2|\nabla^a H^p|^2}\ dx + hY_0|\nabla^s H^p|(\Omega)\,. \end{aligned} \tag{6.4}$$

$\mathcal{H}_{GA}$ is lower semicontinuous with respect to the weak* convergence in BV. We denote by $\mathcal{V}_{\mathcal{H}_{GA}}$, as we have already done for $\mathcal{H}$, the $\mathcal{H}_{GA}$-variation. Arguing as in Proposition 3.2.4, it is possible to show that $\mathcal{V}_{\mathcal{H}_{GA}}$ is weakly* lower semicontinuous.

The free energy in GA is the same that we used in the previous chapters. However, note that in GA

$$\Psi_2(\mathrm{curl}(H^p)) = \frac{\mu L^2}{2} \int_\Omega |\,\mathrm{curl}(H^p)|^2\ dx$$

depends only on the symmetric part of H^p.

Let us fix a boundary displacement w and some forces f, g as in (3.26) and (3.25). We assume the existence of a function ρ as in (3.47) such that (3.48) and (3.50) hold.

An energetic solution for GA is a map

$$t \mapsto (u(t), H^e(t), H^p(t))$$

such that the following properties hold:

- Admissibility:

$$(u(t), H^e(t), H^p(t)) \in \mathcal{A}_{GA}(w(t)) \quad \forall\ t \in [0, T]\,, \tag{6.5}$$

- Global stability:

$$\begin{aligned}\mathcal{E}(t) \le \Psi_1(e) + \Psi_2(\mathrm{curl}(p)) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}_{GA}(p - H^p(t)) \\ \forall\ (v, e, p) \in \mathcal{A}_{GA}(w(t)) \quad \forall\ t \in [0, T]\,,\end{aligned} \tag{6.6}$$

- Bounded variation: H^p has bounded variation as a map

$$H^p : [0, T] \to \mathrm{BV}(\Omega; M^{3\times 3}_{D,sym})\,,$$

- Energy balance:

$$\begin{aligned}\mathcal{E}(t) + \mathcal{V}_{\mathcal{H}_{GA}}(H^p; 0, t) = \mathcal{E}(0) + \int_0^t \int_\Omega \mathbb{C}H^e(\tau) : E\dot{w}(\tau)\ dx d\tau \\ - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\ d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\ d\tau \quad \forall\ t \in [0, T]\,,\end{aligned} \tag{6.7}$$

where

$$\mathcal{E}(t) := \Psi_1(H^e(t)) + \Psi_2(\operatorname{curl}(H^p(t))) - \langle \mathcal{L}(t), u(t)\rangle\,.$$

6.1.2 The convergence result

Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\chi_n \to +\infty$ and let $(u_{0n}, H^e_{0n}, H^p_{0n})$ be a sequence of initial data such that

$$\begin{aligned}
&u_{0n} \rightharpoonup u_0 && \text{in } W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3)\,, && (6.8)\\
&H^e_{0n,sym} \to H^e_0 && \text{in } L^2(\Omega; M^{3\times3}_{sym})\,, && (6.9)\\
&H^p_{0n,sym} \overset{*}{\rightharpoonup} H^p_0 && \text{in } \mathrm{BV}(\Omega; M^{3\times3}_{D,sym})\,, && (6.10)\\
&H^p_{0n,skew} \overset{*}{\rightharpoonup} 0 && \text{in } \mathscr{M}_b(\Omega; M^{3\times3}_{skew})\,, && (6.11)\\
&\operatorname{curl}(H^p_{0n}) \to \operatorname{curl}(H^p_0) && \text{in } L^2(\Omega; M^{3\times3}) && (6.12)
\end{aligned}$$

In particular, we have the convergence of the free energy

$$\Psi(H^e_{0n}, \operatorname{curl}(H^p_{0n})) \to \Psi(H^e_0, \operatorname{curl}(H^p_0))\,. \tag{6.13}$$

For every $n \in \mathbb{N}$ let

$$t \mapsto (u_n(t), H^e_n(t), H^p_n(t))$$

be an energetic solution to the Gurtin model with $\chi = \chi_n$, boundary datum w, and initial datum $(u_{0n}, H^e_{0n}, H^p_{0n})$.

Owing to Theorem 5.2.1 and Corollary 5.2.1, we can apply Theorem 2.1.1 and infer that there exist

$$\begin{aligned}
&u \in \mathrm{AC}(0,T; W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3))\,,\\
&H^e \in \mathrm{AC}(0,T; L^2(\Omega; M^{3\times3}_{sym}))\,,\\
&H^p \in \mathrm{AC}(0,T; \mathrm{BV}(\Omega; M^{3\times3}_{skew}))\,,
\end{aligned}$$

and a subsequence (χ_{n_k}) such that for every $t \in [0,T]$

$$\begin{aligned}
&u_{n_k}(t) \rightharpoonup u(t) && \text{in } W^{1,\frac{3}{2}}(\Omega;\mathbb{R}^3)\,, && (6.14)\\
&H^e_{n_k,sym}(t) \rightharpoonup H^e(t) && \text{in } L^2(\Omega; M^{3\times3}_{sym})\,, && (6.15)\\
&H^p_{n_k,sym}(t) \overset{*}{\rightharpoonup} H^p(t) && \text{in } \mathrm{BV}(\Omega; M^{3\times3}_{D,sym})\,. && (6.16)
\end{aligned}$$

Indeed, the sequences $(u_n)_{n\in\mathbb{N}}, (H^e_{n,sym})_{n\in\mathbb{N}}$, and $(H^p_{n,sym})_{n\in\mathbb{N}}$ are equiabsolutely continuous with respect to n with values in the corresponding spaces. Note that, by construction, the map

$$H^p : [0,T] \to \mathrm{BV}(\Omega; M^{3\times3}_{D,sym})$$

has bounded variation. For every q in the predual of $\mathscr{M}_b(\Omega; M^{3\times3}_{skew})$ we have

$$\begin{aligned}
&|\langle H^p_{n_k,skew}(t), q\rangle| \le \|H^p_{n_k,skew}(t) - H^p_{n_k,skew}(0)\|_{\mathscr{M}_b}\|q\|_{C_0}\\
&+ |\langle H^p_{n_k,skew}(0), q\rangle| \le \mathcal{V}(H^p_{n_k}; 0,t)\|q\|_{C_0} + |\langle H^p_{n_k,skew}(0), q\rangle|\,.
\end{aligned}$$

By Proposition 4.3.1 and Remark 4.3.1 we have

$$\mathcal{V}(H^p_{n_k,skew};0,t) \leq \frac{C_2}{\beta_{n_k}} \to 0\,,$$

where β_{n_k} is defined as in Proposition 4.1.3 with $\chi = \chi_{n_k}$. Therefore, we deduce that for every $t \in [0,T]$

$$H^p_{n_k,skew}(t) \overset{*}{\rightharpoonup} 0 \quad \text{in } \mathscr{M}_b(\Omega; M^{3\times3}_{skew})\,. \tag{6.17}$$

We now show that the map

$$t \mapsto (u(t), H^e(t), H^p(t))$$

is an energetic solution for GA.

Proposition 6.1.1. *The admissibility (6.5) holds for the map*

$$t \mapsto (u(t), H^e(t), H^p(t))\,.$$

Proof. Let $t \in [0,T]$. The sequence $(\mathrm{curl}(H^p_{n_k}(t)))_k$ is uniformly bounded with respect to k owing to Proposition 4.3.1 and Remark 4.3.1. Hence, by Lemma 4.1.1 and convergences (6.16)-(6.17) we have

$$\mathrm{curl}(H^p_{n_k}(t)) \rightharpoonup \mathrm{curl}(H^p(t)) \quad \text{in } L^2(\Omega; M^{3\times3}) \quad \forall\, t \in [0,T]\,. \tag{6.18}$$

Therefore, condition (6.1) hold. By the decomposition

$$Eu_{n_k}(t) = H^e_{n_k,sym}(t) + H^p_{n_k,sym}(t) \qquad \forall\, k \in \mathbb{N}$$

passing to the limit we obtain (6.2). Finally, (6.3) holds trivially. ■

For the rest of the section we denote by $\mathcal{H}_n$ the plastic dissipation for the Gurtin model with $\chi = \chi_n$ and by $\mathcal{E}_n$ the functional $\mathcal{E}$ defined as in (3.58) for the map

$$t \mapsto (u_n(t), H^e_n(t), H^p_n(t))\,.$$

Proposition 6.1.2. *The global stability condition (6.6) holds for the map*

$$t \mapsto (u(t), H^e(t), H^p(t))\,.$$

Proof. Let $t \in [0,T]$ and $(v,e,p) \in \mathcal{A}_{GA}(0)$. The triplet $(v, e + (\nabla v - Ev), p)$ belongs to $\mathcal{A}(0)$. Hence by Lemma 3.2.1, for every $k \in \mathbb{N}$

$$\left| \int_\Omega \mathbb{C} H^e_{n_k,sym}(t) : e\; dx + \mu L^2 \int_\Omega \mathrm{curl}(H^p_{n_k}(t)) : \mathrm{curl}(p)\; dx - \langle \mathcal{L}(t), v\rangle \right| \leq \mathcal{H}_{n_k}(p)\,.$$

Since p takes values in the space of symmetric matrices, we have $\mathcal{H}_{n_k}(p) = \mathcal{H}_{GA}(p)$. Therefore, by convergences (6.15) and (6.18), passing to the limit as $k \to \infty$ we obtain

$$\left| \int_\Omega \mathbb{C} H^e(t) : e\; dx + \mu L^2 \int_\Omega \mathrm{curl}(H^p(t)) : \mathrm{curl}(p)\; dx - \langle \mathcal{L}(t), v\rangle \right|$$

$$\leq \mathcal{H}_{GA}(p)\,. \tag{6.19}$$

Consider the convex functional

$$\mathcal{F}(v,e,p) := \Psi_1(e) + \Psi_2(\mathrm{curl}(p)) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}_{GA}(p - H^p(t))$$

defined on the convex set $\mathcal{A}_{GA}(w(t))$. Now let $(v,e,p) \in \mathcal{A}_{GA}(w(t))$ and $\lambda \in (0,1]$. By convexity we obtain

$$\begin{aligned}
&\mathcal{F}(v,e,p) - \mathcal{F}(u(t),H^e(t),H^p(t)) = \frac{1}{\lambda}[\mathcal{F}(u(t),H^e(t),H^p(t))\\
&+ \lambda(\mathcal{F}(v,e,p) - \mathcal{F}(u(t),H^e(t),H^p(t))) - \mathcal{F}(u(t),H^e(t),H^p(t))]\\
&\geq \frac{1}{\lambda}[\mathcal{F}(u(t)+\lambda(v-u(t)), H^e(t)+\lambda(e-H^e(t)), H^p(t)+\lambda(p-H^p(t)))\\
&- \mathcal{F}(u(t),H^e(t),H^p(t))] = \lambda\Psi_1(e-H^e(t)) + \lambda\Psi_2(\mathrm{curl}(p-H^p(t)))\\
&+ \int_\Omega \mathbb{C}H^e(t) : (e-H^e(t))\ dx + \mu L^2 \int_\Omega \mathrm{curl}(H^p(t)) : (\mathrm{curl}(p-H^p(t)))\ dx\\
&- \langle \mathcal{L}(t), v-u(t)\rangle + \mathcal{H}_{GA}(p-H^p(t))
\end{aligned} \tag{6.20}$$

Note that the triplet

$$(v-u(t), e-H^e(t), p-H^p(t))$$

belongs to $\mathcal{A}_{GA}(0)$, hence by (6.19) the right-hand side of (6.20) is nonnegative. Therefore, we have proved that

$$\mathcal{F}(v,e,p) - \mathcal{F}(u(t),H^e(t),H^p(t)) \geq 0 \quad \forall\ (v,e,p) \in \mathcal{A}_{GA}(w(t))$$

that is, the global stability condition (6.6). ∎

Proposition 6.1.3. *The map*

$$t \mapsto (u(t), H^e(t), H^p(t))$$

satisfies the following energy inequality:

$$\begin{aligned}
\mathcal{E}(t) + \mathcal{V}_{\mathcal{H}_{GA}}(H^p;0,t) &\leq \mathcal{E}(0) + \int_0^t\int_\Omega \mathbb{C}H^e(\tau) : E\dot{w}(\tau)\ dxd\tau\\
&- \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\ d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\ d\tau \quad \forall\ t\in[0,T]\,.
\end{aligned}$$

Proof. Let $t \in [0,T]$. By definition of energetic solution we have

$$\begin{aligned}
\mathcal{E}_{n_k}(t) + \mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k};0,t) &= \mathcal{E}_{n_k}(0) + \int_0^t\int_\Omega \mathbb{C}H^e_{n_k,sym}(\tau) : E\dot{w}(\tau)\ dxd\tau\\
&- \int_0^t \langle \dot{\mathcal{L}}(\tau), u_{n_k}(\tau)\rangle\ d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\ d\tau \quad \forall\ t\in[0,T]\,.
\end{aligned}$$

for every $k \in \mathbb{N}$. By dominated convergence and (6.13)-(6.15), the right-hand side converges to

$$\mathcal{E}(0) + \int_0^t \int_\Omega \mathbb{C}H^e(\tau) : E\dot{w}(\tau)\ dxd\tau - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\ d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\ d\tau$$

Since Ψ_1, Ψ_2, and $\mathcal{L}(t)$ are weakly lower semicontinuous with respect to the natural convergences, we obtain

$$\mathcal{E}(t) \le \liminf_{n\to\infty} \mathcal{E}_{n_k}(t)\,.$$

Let $0 = t_0, \ldots, t_h = t$ be a partition of $[0,t]$. By definition of $\mathcal{H}_{GA}$ we have

$$\begin{aligned}\mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k}; 0, t) &\ge \sum_{i=1}^h \mathcal{H}_{n_k}(H^p_{n_k}(t_i) - H^p_{n_k}(t_{i-1}))\\ &\ge \sum_{i=1}^h \mathcal{H}_{GA}(H^p_{n_k,sym}(t_i) - H^p_{n_k,sym}(t_{i-1}))\,.\end{aligned}$$

Passing to the supremum over all partitions of the interval $[0,t]$ we obtain by the weak* lower semicontinuity of $\mathcal{V}_{\mathcal{H}_{GA}}$

$$\mathcal{V}_{\mathcal{H}_{GA}}(H^p; 0, t) \le \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}_{GA}}(H^p_{n_k,sym}; 0, t) \le \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k}; 0, t)\,.$$

Combining all these results together the proof is concluded. ■

Finally, arguing as in Lemma 7.3 in [6], the converse energy inequality can be proved. Therefore, the map

$$t \mapsto (u(t), H^e(t), H^p(t))$$

is an energetic solution for the GA model.

6.1.3 A few considerations

The Gurtin-Anand model is obtained from the Gurtin model assuming that the body is plastically irrotational, i.e., the plastic spin is null. For this reason we have to assume condition (6.11) on the initial data. With this assumption we fully recover the energetic formulation given by Giacomini and Lussardi in [6]. If, instead, we assume that

$$H^p_{0n,skew} \overset{*}{\rightharpoonup} W_0 \quad \text{in } \mathscr{M}_b(\Omega; M^{3\times 3}_{skew}) \tag{6.21}$$

with $W_0 \not\equiv 0$, the same argument as in the previous section shows that

$$H^p_{n_k,skew}(t) \overset{*}{\rightharpoonup} W_0 \quad \text{in } \mathscr{M}_b(\Omega; M^{3\times 3}_{skew}) \quad \forall\, t \in [0,T]\,.$$

Therefore, the body has constant, but nonzero, plastic spin. At the limit one obtains a free energy of the form

$$\Psi(H^e, H^p) = \frac{1}{2}\int_\Omega \mathbb{C}H^e : H^e \; dx + \frac{\mu L^2}{2}\int_\Omega \mathrm{curl}(H^p + W_0) \; dx \,. \tag{6.22}$$

In particular, we do not recover a solution in the framework given by Giacomini and Lussardi. However, one could modify the notion of admissible triplets and the definition of free energy for the GA model accounting for a fixed and constant plastic spin W_0. More precisely, we could require

$$\mathrm{curl}(H^p + W_0) \in L^2(\Omega; M^{3\times3})$$

instead of

$$\mathrm{curl}(H^p) \in L^2(\Omega; M^{3\times3})$$

in (6.1) and define the free energy as in (6.22). The model obtained should still satisfy the same flow rule. Moreover, we expect to retrieve a solution of such model when $\chi \to +\infty$ if we assume (6.21) in place of (6.11).

We conclude the section by observing that for every initial datum (u_0, H_0^e, H_0^p) for the GA model it is always possible to construct a sequence $(u_{0n}, H_{0n}^e, H_{0n}^p)_n$ of initial data for the Gurtin model that satisfies convergences (6.8)-(6.12). Indeed, one can simply take

$$(u_{0n}, H_{0n}^e, H_{0n}^p) := (u_0, H_0^e + (\nabla u_0 - Eu_0), H_0^p)\,.$$

6.2 Asymptotic behavior as $h, L \to 0$

In [6] the authors show the convergence of energetic solutions to the GA model to energetic solutions to the Prandtl-Reuss model of perfect plasticity studied in [15] by Dal Maso, DeSimone, and Mora. In this section we aim at extending this result to sequences of energetic solutions to the Gurtin model.

We will assume in this section that Ω and Γ are of class C^2 where Γ is the shared boundary of Γ_D and Γ_N (with respect to the relative topology on $\partial\Omega$).

6.2.1 The Prandtl-Reuss model

We briefly introduce the Prandtl-Reuss model, henceforth abbreviated as PR. As in the GA model, there is no dependence on the skew-symmetric part of the strains.

Given a boundary value $z \in H^1(\Omega; \mathbb{R}^3)$ we will say that a triplet (u, H^e, H^p) is admissible for the boundary value z in the PR model if and only if

$$u \in \mathrm{BD}(\Omega)\,, \qquad H^e \in L^2(\Omega; M^{3\times3}_{sym})\,,$$
$$H^p \in \mathscr{M}_b(\Omega \cup \Gamma_D; M^{3\times3}_{D,sym})\,, \tag{6.23}$$

$$Eu = H^e + H^p\,, \tag{6.24}$$

$$H^p = (w - u) \odot \nu \mathscr{H}^2 \text{ on } \Gamma_D\,. \tag{6.25}$$

Here $\mathrm{BD}(\Omega)$ is the space of functions with bounded deformation (see Section 2.1) and ν is the outer normal vector to $\partial\Omega$. We will denote by $\mathcal{A}_{PR}(z)$ the set of admissible triplets for the boundary value z in the PR model.

The plastic dissipation functional is defined once again using the notion of convex functions of measure introduced in [9]. More precisely,

$$\mathcal{H}_{PR} : \mathscr{M}_b(\Omega \cup \Gamma_D; M^{3\times 3}_{D,sym}) \to \mathbb{R} : \quad H^p \mapsto Y_0 \int_{\Omega\cup\Gamma_D} \left| \frac{H^p}{|H^p|} \right| \, d|H^p|\,, \tag{6.26}$$

where $|H^p|$ is the total variation of H^p and $\dfrac{H^p}{|H^p|}$ is the Radon-Nikodým derivative of H^p with respect to $|H^p|$. In particular, if H^p is absolutely continuous with respect to $\mathscr{L}^3$, then

$$\mathcal{H}_{PR}(H^p) = Y_0 \int_\Omega |H^p| \, dx\,.$$

$\mathcal{H}_{PR}$ is lower semicontinuous with respect to the weak* convergence in $\mathscr{M}_b(\Omega \cup \Gamma_D; M^{3\times 3}_{sym})$ by construction (see Theorem 3 in [9]). As a result, the $\mathcal{H}_{PR}$-variation, denoted by $\mathcal{V}_{\mathcal{H}_{PR}}$, is weakly* lower semicontinuous, too.

In the PR model the free energy does not depend on the Burgers vector and reduces to Ψ_1.

Let us fix a boundary displacement w as in (3.26) and the forces

$$f \in \mathrm{AC}(0,T; L^3(\Omega;\mathbb{R}^3))\,, \;\; g \in \mathrm{AC}(0,T; L^\infty(\Gamma_N;\mathbb{R}^3))\,. \tag{6.27}$$

Here we need to assume that g takes values in $L^\infty(\Gamma_N;\mathbb{R}^3)$ since the trace of admissible displacements belongs to $L^1(\partial\Omega;\mathbb{R}^3)$. We assume the existence of a function ρ as in (3.47) such that (3.48) and (3.50) hold.

An energetic solution for the PR model is a map

$$t \mapsto (u(t), H^e(t), H^p(t))$$

such that the following properties hold:

- Admissibility:

$$(u(t), H^e(t), H^p(t)) \in \mathcal{A}_{PR}(w(t)) \quad \forall\, t \in [0,T]\,, \tag{6.28}$$

- Global stability:

$$\mathcal{E}_{PR}(t) \le \Psi_1(e) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}_{PR}(p - H^p(t)) \qquad \forall\, (v,e,p) \in \mathcal{A}_{PR}(w(t)) \quad \forall\, t \in [0,T]\,, \tag{6.29}$$

- Bounded variation: H^p has bounded variation as a map

$$H^p : [0,T] \to \mathscr{M}_b(\Omega \cup \Gamma_D; M^{3\times 3}_{D,sym})\,,$$

- Energy balance:

$$\mathcal{E}_{PR}(t) + \mathcal{V}_{\mathcal{H}_{PR}}(H^p; 0, t) = \mathcal{E}_{PR}(0) + \int_0^t \int_\Omega \mathbb{C}H^e(\tau) : E\dot{w}(\tau)\, dx d\tau$$
$$- \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle\, d\tau \quad \forall\, t \in [0, T]\,, \tag{6.30}$$

where
$$\mathcal{E}_{PR}(t) := \Psi_1(H^e(t)) - \langle \mathcal{L}(t), u(t)\rangle\,.$$

6.2.2 The convergence result

Let (L_n) and (h_n) be sequences such that $h_n, L_n \to 0$. As done for the derivation of the GA model, we assume to have a sequence of initial data $(u_{0n}, H^e_{0n}, H^p_{0n})$ such that

$$u_{0n} \overset{*}{\rightharpoonup} u_0 \qquad \text{in } \mathrm{BD}(\Omega; \mathbb{R}^3)\,, \tag{6.31}$$
$$H^e_{0n,sym} \to H^e_0 \quad \text{in } L^2(\Omega; M^{3\times 3}_{sym})\,, \tag{6.32}$$
$$H^p_{0n,sym} \overset{*}{\rightharpoonup} H^p_0 \quad \text{in } \mathscr{M}_b(\Omega; M^{3\times 3}_{D,sym})\,. \tag{6.33}$$

Moreover, we assume that the sequence $(\mathrm{curl}(H^p_{0n}))_n$ is bounded in $L^2(\Omega; M^{3\times 3})$. Hence, we deduce the convergence of the free energy

$$\Psi_1(H^e_{0n}) + \frac{\mu L_n^2}{2} \int_\Omega |\,\mathrm{curl}(H^p_{0n})|^2\, dx \to \Psi_1(H^e_0)\,, \tag{6.34}$$

For every $n \in \mathbb{N}$ let
$$t \mapsto (u_n(t), H^e_n(t), H^p_n(t))$$
be an energetic solution to the Gurtin model with $h = h_n$, $L = L_n$, and initial datum $(u_{0n}, H^e_{0n}, H^p_{0n})$. The next lemma grants the existence of a suitable converging subsequence of energetic solutions.

Lemma 6.2.1. *There exist a subsequence (n_k) and two functions*

$$u \in \mathrm{AC}(0, T; \mathrm{BD}(\Omega))$$
$$H^e \in \mathrm{AC}(0, T; L^2(\Omega; M^{3\times 3}_{sym}))$$

such that for every $t \in [0, T]$

$$u_{n_k}(t) \overset{*}{\rightharpoonup} u(t) \qquad \textit{in } \mathrm{BD}(\Omega; \mathbb{R}^3)\,, \tag{6.35}$$
$$H^e_{n_k,sym}(t) \rightharpoonup H^e(t) \quad \textit{in } L^2(\Omega; M^{3\times 3}_{sym})\,. \tag{6.36}$$

Morevoer, there exists a function $H^p \in \mathrm{AC}(0, T; \mathscr{M}_b(\Omega \cup \Gamma_D; M^{3\times 3}_{D,sym}))$ such that for every $t \in [0, T]$, by setting $H^p_{n,sym}(t) = 0$ on Γ_D for every $n \in \mathbb{N}$, we have
$$H^p_{n_k,sym}(t) \overset{*}{\rightharpoonup} H^p(t) \quad \textit{in } \mathscr{M}_b(\Omega \cup \Gamma_D; M^{3\times 3}_{D,sym})\,. \tag{6.37}$$
Finally, for every $t \in [0, T]$ the triplet $(u(t), H^e(t), H^p(t))$ is admissible for the

boundary value $w(t)$ in the PR model.

Proof. Since Γ_D is open in $\partial\Omega$ there exists $U \subset \mathbb{R}^3$ open such that $\Gamma_D = \partial\Omega \cap U$. Let $\widetilde{\Omega} := \Omega \cup U$. We define

$$\widetilde{u}_n(t) = \begin{cases} u(t) & \text{on } \Omega \\ w(t) & \text{on } \widetilde{\Omega}\backslash\Omega \end{cases} \qquad \widetilde{H}^e_{n,sym}(t) = \begin{cases} H^e_{n,sym}(t) & \text{on } \Omega \\ Ew(t) & \text{on } \widetilde{\Omega}\backslash\Omega \end{cases}$$

$$\widetilde{H}^p_{n,sym}(t) = \begin{cases} H^p_{n,sym}(t) & \text{on } \Omega \\ 0 & \text{on } \widetilde{\Omega}\backslash\Omega \end{cases}$$

By Corollary 5.2.1 and convergences (6.31)-(6.34), the sequences $(\widetilde{H}^e_{n,sym})$ and $(\widetilde{H}^p_{n,sym})$ are equi-absolutely continuous with respect to n as maps with values in $L^2(\widetilde{\Omega}; M^{3\times3}_{sym})$ and $L^1(\widetilde{\Omega}; M^{3\times3}_{D,sym})$, respectively. Observe that

$$E\widetilde{u}_n(t) = \widetilde{H}^e_{n,sym}(t) + \widetilde{H}^p_{n,sym}(t) \quad \text{on } \widetilde{\Omega}\,. \tag{6.38}$$

Therefore, by Theorem 2.2.5 $(\widetilde{u}_n)$ is equi-absolutely continuous with respect to n as maps with values in $\mathrm{BD}(\widetilde{\Omega}; \mathbb{R}^3)$. Applying Theorem 2.1.1 we deduce that there exists a subsequence (n_k) and maps

$$\begin{aligned} &\widetilde{u} \in \mathrm{AC}(0, T; \mathrm{BD}(\widetilde{\Omega}; \mathbb{R}^3))\,, \\ &\widetilde{H}^e \in \mathrm{AC}(0, T; L^2(\widetilde{\Omega}; M^{3\times3}_{sym}))\,, \\ &\widetilde{H}^p \in \mathrm{AC}(0, T; \mathscr{M}_b(\widetilde{\Omega}; M^{3\times3}_{D,sym}))\,, \end{aligned}$$

such that for every $t \in [0, T]$

$$\begin{aligned} &\widetilde{u}_{n_k}(t) \overset{*}{\rightharpoonup} \widetilde{u}(t) && \text{in } \mathrm{BD}(\widetilde{\Omega}; \mathbb{R}^3)\,, \\ &\widetilde{H}^e_{n_k,sym}(t) \rightharpoonup \widetilde{H}^e(t) && \text{in } L^2(\widetilde{\Omega}; M^{3\times3}_{sym})\,, \\ &\widetilde{H}^p_{n,sym}(t) \overset{*}{\rightharpoonup} \widetilde{H}^p(t) && \text{in } \mathscr{M}_b(\widetilde{\Omega}; M^{3\times3}_{D,sym})\,. \end{aligned}$$

We define u and H^e as the restriction of $\widetilde{u}$ and $\widetilde{H}^e$, respectively, to Ω. Similarly, we define H^p as the restriction of $\widetilde{H}^p$ to $\Omega \cup \Gamma_D$. Therefore, convergences (6.35) and (6.36) follow immediately. On $\widetilde{\Omega}\backslash\overline{\Omega}$ it must be $\widetilde{H}^p = 0$, thus, convergence (6.37) holds. Let $t \in [0, T]$. Passing to the limit in (6.38) we obtain

$$E\widetilde{u}(t) = \widetilde{H}^e(t) + \widetilde{H}^p(t) \quad \text{on } \widetilde{\Omega}\,.$$

Since on $\widetilde{\Omega}\backslash\overline{\Omega}$ it must be $\widetilde{u}(t) = w(t)$ and $\widetilde{H}^e(t) = Ew(t)$ we deduce that on Γ_D

$$\widetilde{H}^p(t) = E\widetilde{u}(t) = (w(t) - u(t)) \odot \nu \mathscr{H}^2\,.$$

This proves that the triplet $(u(t), H^e(t), H^p(t))$ is admissible for the boundary value $w(t)$ in the PR model. ∎

Let u, H^e, and H^p be the maps given in Lemma 6.2.1. We are now in a position to prove that the map

$$t \mapsto (u(t), H^e(t), H^p(t)),$$

is an energetic solution to the PR model. By the previous lemma we only need to prove the global stability (6.29) and the energy balance (6.30).

Proposition 6.2.1. *The map*

$$t \mapsto (u(t), H^e(t), H^p(t))$$

satisfies the global stability (6.29).

Proof. For every $t \in [0, T]$ let $T(t) = \mathbb{C}H^e(t)$. In view of Theorem 3.6 in [15] it is sufficient to prove that for every $t \in [0, T]$

$$\begin{cases} -\operatorname{div} T(t) = f(t) & \text{in } L^2(\Omega; \mathbb{R}^3), \\ \gamma_\nu(T(t)) = g(t) & \text{on } \Gamma_N, \end{cases}$$

and $|T_D(t)| \leq Y_0$ almost everywhere in Ω. The condition on the normal trace has to be intended as stated in Proposition 5.1.1. Let $t \in [0, T]$ and let $T_{n_k}(t) = \mathbb{C}H^e_{n_k,sym}(t)$. By Proposition 5.1.1, for every $k \in \mathbb{N}$ we have

$$\begin{cases} -\operatorname{div} T_{n_k}(t) = f(t) & \text{in } L^2(\Omega; \mathbb{R}^3), \\ \gamma_\nu(T_{n_k}(t)) = g(t) & \text{on } \Gamma_N. \end{cases}$$

By convergence (6.36) we deduce

$$T_{n_k}(t) \rightharpoonup T(t) \quad \text{in } L^2(\Omega; M^{3\times3}_{sym}). \tag{6.39}$$

By a density argument similar to the one used in Lemma 4.1.1 it is possible to prove that $\operatorname{div} T_{n_k}(t) \rightharpoonup \operatorname{div} T(t)$, hence,

$$-\operatorname{div} T(t) = f(t) \quad \text{in } L^2(\Omega; \mathbb{R}^3).$$

In particular, by the definition of γ_ν, we infer that $\gamma_\nu(T(t)) = g(t)$ on Γ_N. Let $T^p_{n_k}$, $K_{n_k,diss}$, $K_{n_k,en}$, and $R_{n_k}(t)$ be the plastic stresses given by Proposition 5.1.2, (5.10), and (5.1) for the energetic solution

$$t \mapsto (u_{n_k}(t), H^e_{n_k}(t), H^p_{n_k}(t)).$$

By Proposition 4.3.1 and Remark 4.3.1 we have

$$\|R_{n_k}(t)\|_{L^2} = \mu L^2_{n_k} \|\operatorname{curl}(H^p_{n_k}(t))\|_{L^2} \leq 2L_{n_k}\sqrt{C_2\mu} \to 0.$$

Therefore, $K_{n_k,en}(t) \to 0$ in $L^2(\Omega; M^{3\times3})$. Since by Proposition 5.2.1 the constraint

$$\sqrt{|T^p_{n_k,sym}(t)|^2 + \frac{1}{h^2_{n_k}}|K_{n_k,diss}(t)|^2} \leq Y_0 \quad \text{a.e. in } \Omega \tag{6.40}$$

holds for every $k \in \mathbb{N}$, we deduce that

$$K_{n_k,diss}(t) \to 0 \quad \text{in } L^\infty(\Omega; M_D^{3\times3\times3})\,.$$

Hence, $K_{n_k}(t) \to 0$ in $L^2(\Omega; M_D^{3\times3\times3})$. By Proposition 5.1.3 we have that

$$T^p_{n_k,sym}(t) = T_{n_k,D}(t) + (\operatorname{div}(K_{n_k}(t)))_{sym} \quad \text{in } L^2(\Omega; M_D^{3\times3}) \quad \forall\, k \in \mathbb{N}\,. \tag{6.41}$$

Combining (6.40) and (6.41) we deduce that the sequence $((\operatorname{div} K_{n_k}(t))_{sym})_k$ is bounded in $L^2(\Omega; M^{3\times3})$, thus

$$(\operatorname{div} K_{k_n}(t))_{sym} \rightharpoonup 0 \quad \text{in } L^2(\Omega; M^{3\times3})\,. \tag{6.42}$$

Finally, by (6.41) we infer

$$T^p_{n_k,sym}(t) \rightharpoonup T_D(t) \quad \text{in } L^2(\Omega; M_D^{3\times3})\,. \tag{6.43}$$

We define

$$\mathcal{K} = \left\{ A \in L^2(\Omega; M_D^{3\times3}) \ :\ |A_{sym}| \le Y_0 \text{ a.e in } \Omega \right\}\,.$$

Since by Proposition 5.2.1 $T^p_{n_k}(t) \in \mathcal{K}$ for every $k \in \mathbb{N}$ and $\mathcal{K}$ is closed with respect to weak convergence, we deduce that $T_D(t) \in \mathcal{K}$ concluding the proof. ■

Before introducing the next proposition it is important to recall that the weak* convergence in $\mathrm{BD}(\Omega;\mathbb{R}^3)$ is sufficient to guarantee convergence of the volume force term, but not of the surface force term. Indeed, the trace operator

$$\gamma : \mathrm{BD}(\Omega;\mathbb{R}^3) \to L^1(\partial\Omega;\mathbb{R}^3)$$

is not continuous if we equip $\mathrm{BD}(\Omega;\mathbb{R}^3)$ of the weak* topology. The safe-load condition will be crucial to overcome this problem.

Proposition 6.2.2. *The map*

$$t \mapsto (u(t), H^e(t), H^p(t))$$

satisfies the following energy inequality:

$$\mathcal{E}_{PR}(t) + \mathcal{V}_{\mathcal{H}_{PR}}(H^p; 0, t) \le \mathcal{E}_{PR}(0) + \int_0^t \int_\Omega \mathbb{C}H^e(\tau) : E\dot{w}(\tau)\; dx d\tau \\ - \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle \; d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle \; d\tau \quad \forall\, t \in [0,T]\,.$$

Proof. By definition of energetic solution, we have

$$\mathcal{E}_{n_k}(t) + \mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k}; 0, t) = \mathcal{E}_{n_k}(0) + \int_0^t \int_\Omega \mathbb{C}H^e_{n_k,sym}(\tau) : E\dot{w}(\tau)\; dx d\tau \\ - \int_0^t \langle \dot{\mathcal{L}}(\tau), u_{n_k}(\tau)\rangle \; d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau)\rangle \; d\tau \quad \forall\, t \in [0,T] \tag{6.44}$$

for every $k \in \mathbb{N}$, where

$$\mathcal{E}_{n_k}(t) = \Psi_1(H^e_{n_k,sym}(t)) + \frac{\mu L^2_{n_k}}{2} \int_\Omega |\operatorname{curl}(H^p_{n_k}(t))|^2 \, dx - \langle \mathcal{L}(t), u_{n_k}(t) \rangle$$

and $\mathcal{H}_{n_k}$ is the plastic dissipation for the Gurtin model with $h = h_{n_k}$.
Let $t \in [0, T]$. Integrating by parts, (6.44) can be rewritten in the following form:

$$\begin{aligned}
&\Psi_1(H^e_{n_k,sym}(t)) + \frac{\mu L^2_{n_k}}{2} \int_\Omega |\operatorname{curl}(H^p_{n_k}(t))|^2 \, dx + \mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k}; 0, t) \\
&- \int_0^t \langle \mathcal{L}(\tau), \dot{u}_{n_k}(\tau) \rangle \, d\tau = \Psi_1(H^e_{n_k,sym}(0)) + \frac{\mu L^2_{n_k}}{2} \int_\Omega |\operatorname{curl}(H^p_{n_k}(0))|^2 \, dx \\
&+ \int_0^t \int_\Omega \mathbb{C} H^e_{n_k,sym}(\tau) : E\dot{w}(\tau) \, dx d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau) \rangle \, d\tau \, . \qquad (6.45)
\end{aligned}$$

Owing to convergences (6.34) and (6.36), the right-hand side of (6.45) converges by dominated convergence to

$$\Psi_1(H^e(0)) + \int_0^t \int_\Omega \mathbb{C} H^e(\tau) : E\dot{w}(\tau) \, dx d\tau - \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau) \rangle \, d\tau \, .$$

By convexity Ψ_1 is weakly lower semicontinuous, hence,

$$\Psi_1(H^e(t)) \le \liminf_{k \to \infty} \left[\Psi_1(H^e_{n_k,sym}(t)) + \frac{\mu L^2_{n_k}}{2} \int_\Omega |\operatorname{curl}(H^p_{n_k}(t))|^2 \, dx \right] .$$

To conclude it is sufficient to prove that

$$\begin{aligned}
&\liminf_{k \to \infty} \left[\mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k}; 0, t) - \int_0^t \langle \mathcal{L}(\tau), \dot{u}_{n_k}(\tau) \rangle \, d\tau \right] \\
&\ge \mathcal{V}_{\mathcal{H}_{PR}}(H^p; 0, t) - \int_0^t \langle \mathcal{L}(\tau), \dot{u}(\tau) \rangle \, d\tau \\
&= \mathcal{V}_{\mathcal{H}_{PR}}(H^p; 0, t) + \langle \mathcal{L}(0), u(0) \rangle - \langle \mathcal{L}(t), u(t) \rangle + \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau) \rangle \, d\tau \, .
\end{aligned}$$

Applying Theorem 7.1 in [15] we write

$$\mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k}; 0, t) = \int_0^t \mathcal{H}_{n_k}(\dot{H}^p_{n_k}(\tau)) \, d\tau \ge \int_0^t \int_\Omega Y_0 |\dot{H}^p_{n_k,sym}(\tau)| \, dx d\tau \, . \qquad (6.46)$$

By Proposition 3.2.5 we have

$$\begin{aligned}
&\int_0^t \langle \mathcal{L}(\tau), \dot{u}_{n_k}(\tau) \rangle \, d\tau = \int_0^t \langle \mathcal{L}(\tau), \dot{w}(\tau) \rangle \, d\tau - \int_0^t \int_\Omega \rho(\tau) : E\dot{w}(\tau) \, dx d\tau \\
&+ \int_0^t \int_\Omega \rho(\tau) : \dot{H}^e_{n_k,sym}(\tau) \, dx d\tau + \int_0^t \int_\Omega \rho_D(\tau) : \dot{H}^p_{n_k,sym}(\tau) \, dx d\tau \\
&= \langle \mathcal{L}(t), w(t) \rangle - \langle \mathcal{L}(0), w(0) \rangle - \int_0^t \langle \dot{\mathcal{L}}(\tau), w(\tau) \rangle \, d\tau + \int_\Omega \rho(0) : Ew(0) \, dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega \rho(t) : Ew(t)\, dx + \int_0^t \int_\Omega \dot\rho(\tau) : Ew(\tau)\, dx + \int_\Omega \rho(t) : H^e_{n_k,sym}(t)\, dx \\
&- \int_\Omega \rho(0) : H^e_{n_k,sym}(0)\, dx - \int_0^t \int_\Omega \dot\rho(\tau) : H^e_{n_k,sym}(\tau)\, dx \\
&+ \int_0^t \int_\Omega \rho_D(\tau) : \dot H^p_{n_k,sym}(\tau)\, dx d\tau\,. \qquad (6.47)
\end{aligned}
$$

Combining (6.46) and (6.47), by the convergence (6.36) we obtain

$$
\begin{aligned}
&\liminf_{k\to\infty} \left[\mathcal{V}_{\mathcal{H}_{n_k}}(H^p_{n_k}; 0, t) - \int_0^t \langle \mathcal{L}(\tau), \dot u_{n_k}(\tau)\rangle\, d\tau \right] \\
&\geq \liminf_{k\to\infty} \int_0^t \int_\Omega \left[Y_0 |\dot H^p_{n_k,sym}(\tau)| - \rho_D(\tau) : \dot H^p_{n_k,sym}(\tau) \right] dx d\tau \\
&- \langle \mathcal{L}(t), w(t)\rangle + \langle \mathcal{L}(0), w(0)\rangle + \int_0^t \langle \dot{\mathcal{L}}(\tau), w(\tau)\rangle\, d\tau - \int_\Omega \rho(0) : Ew(0)\, dx \\
&+ \int_\Omega \rho(t) : Ew(t)\, dx - \int_0^t \int_\Omega \dot\rho(\tau) : Ew(\tau)\, dx - \int_\Omega \rho(t) : H^e(t)\, dx \\
&+ \int_\Omega \rho(0) : H^e(0)\, dx + \int_0^t \int_\Omega \dot\rho(\tau) : H^e(\tau)\, dx\,. \qquad (6.48)
\end{aligned}
$$

Owing to Proposition 2.2 in [15] we rewrite the right-hand side of (6.48) as follows:

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_0^t \int_\Omega \left[Y_0 |\dot H^p_{n_k,sym}(\tau)| - \rho_D(\tau) : \dot H^p_{n_k,sym}(\tau) \right] dx d\tau \\
&+ \langle \mathcal{L}(0), u(0)\rangle - \langle \mathcal{L}(t), u(t)\rangle + \int_0^t \langle \dot{\mathcal{L}}(\tau), u(\tau)\rangle\, d\tau - [\rho_D(0) : H^p(0)](\Omega \cup \Gamma_D) \\
&+ [\rho_D(t) : H^p(t)](\Omega \cup \Gamma_D) - \int_0^t [\dot\rho_D(\tau) : H^p(\tau)](\Omega \cup \Gamma_D)\, d\tau\,, \qquad (6.49)
\end{aligned}
$$

where $[\rho_D(t) : H^p(t)]$ is the measure defined in section 2 of [15]. Therefore, the proof is concluded if we show

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_0^t \int_\Omega \left[Y_0 |\dot H^p_{n_k,sym}(\tau)| - \rho_D(\tau) : \dot H^p_{n_k,sym}(\tau) \right] dx d\tau \\
&\geq \mathcal{V}_{\mathcal{H}_{PR}}(H^p; 0, t) + [\rho_D(0) : H^p(0)](\Omega \cup \Gamma_D) \\
&- [\rho_D(t) : H^p(t)](\Omega \cup \Gamma_D) + \int_0^t [\dot\rho_D(\tau) : H^p(\tau)](\Omega \cup \Gamma_D)\, d\tau\,. \qquad (6.50)
\end{aligned}
$$

Let $\varphi \in C^1(\overline{\Omega})$ be a function such that $0 \leq \varphi \leq 1$ and $\varphi = 0$ on Γ_N. By (3.50) we have that $|\rho_D(\tau)| \leq Y_0$ almost everywhere in Ω for every $\tau \in [0, t]$. Hence, the function

$$
Y_0 |\dot H^p_{n_k,sym}(\tau)| - \rho_D(\tau) : \dot H^p_{n_k,sym}(\tau)
$$

is positive almost everywhere in Ω for every $\tau \in [0, t]$. Therefore, applying once again Theorem 7.1 in [15] we deduce

$$
\int_0^t \int_\Omega \left[Y_0 |\dot H^p_{n_k,sym}(\tau)| - \rho_D(\tau) : \dot H^p_{n_k,sym}(\tau) \right] dx d\tau
$$

$$\geq \int_0^t \int_\Omega \left[Y_0 |\varphi \dot{H}^p_{n_k,sym}(\tau)| - \rho_D(\tau) : \varphi \dot{H}^p_{n_k,sym}(\tau) \right] \, dx d\tau$$

$$= \mathcal{V}_{\mathcal{H}_{PR}}(\varphi H^p_{n_k,sym}; 0, t) - \int_0^t \int_\Omega \rho_D(\tau) : \varphi \dot{H}^p_{n_k,sym}(\tau) \, dx d\tau \,. \tag{6.51}$$

Passing to the limit in (6.51) we obtain, by the lower semicontinuity of the $\mathcal{H}_{PR}$-variation

$$\liminf_{k\to\infty} \int_0^t \int_\Omega \left[Y_0 |\dot{H}^p_{n_k,sym}(\tau)| - \rho_D(\tau) : \dot{H}^p_{n_k,sym}(\tau) \right] \, dx d\tau$$

$$\geq \mathcal{V}_{\mathcal{H}_{PR}}(\varphi H^p; 0, t) + \liminf_{k\to\infty} \left[- \int_0^t \int_\Omega \rho_D(\tau) : \varphi \dot{H}^p_{n_k,sym}(\tau) \, dx d\tau \right] \,. \tag{6.52}$$

Integrating by parts we write

$$- \int_0^t \int_\Omega \rho_D(\tau) : \varphi \dot{H}^p_{n_k,sym}(\tau) \, dx d\tau = - \int_\Omega \rho_D(t) : \varphi H^p_{n_k,sym}(t) \, dx$$

$$+ \int_\Omega \rho_D(0) : \varphi H^p_{n_k,sym}(0) \, dx + \int_0^t \int_\Omega \dot{\rho}_D(\tau) : \varphi H^p_{n_k,sym}(\tau) \, dx d\tau \,. \tag{6.53}$$

Since $W^{1,\frac{3}{2}}(\Omega, \mathbb{R}^3)$ is continuously embedded in $L^2(\Omega; \mathbb{R}^3)$, for every $k \in \mathbb{N}$ and for every $\tau \in [0, t]$

$$(\varphi u_{n_k}(\tau), \varphi H^e_{n_k}(\tau) + \nabla\varphi \otimes u(\tau), \varphi H^p_{n_k}(\tau)) \in \mathcal{A}(\varphi w(t)) \,.$$

Hence, by Proposition 3.2.6 we have

$$\int_0^t \int_\Omega \dot{\rho}_D(\tau) : \varphi H^p_{n_k,sym}(\tau) \, dx d\tau = - \int_0^t \langle \dot{\mathcal{L}}(\tau), \varphi w(\tau) \rangle \, d\tau$$

$$+ \int_0^t \int_\Omega \dot{\rho}(\tau) : \varphi E w(\tau) \, dx d\tau + \int_0^t \int_\Omega \dot{\rho}(\tau) : (\nabla\varphi \odot w(\tau)) \, dx d\tau$$

$$- \int_0^t \int_\Omega \dot{\rho}(\tau) : \varphi H^e_{n_k,sym}(\tau) \, dx d\tau - \int_0^t \int_\Omega \dot{\rho}(\tau) : (\nabla\varphi \odot u_{n_k}(\tau)) \, dx d\tau$$

$$+ \int_0^t \langle \dot{\mathcal{L}}(\tau), \varphi u_{n_k}(\tau) \rangle \, d\tau \,. \tag{6.54}$$

Similarly, applying Proposition 3.2.5 we deduce

$$\int_\Omega \rho_D(0) : \varphi H^p_{n_k,sym}(0) \, dx = -\langle \mathcal{L}(0), \varphi w(0) \rangle + \int_\Omega \rho(0) : \varphi E w(0) \, dx$$

$$+ \int_\Omega \rho(0) : (\nabla\varphi \odot w(0)) \, dx - \int_\Omega \rho(0) : \varphi H^e_{n_k,sym}(0) \, dx$$

$$- \int_\Omega \rho(0) : (\nabla\varphi \odot u_{n_k}(0)) \, dx + \langle \mathcal{L}(0), \varphi u_{n_k}(0) \rangle \tag{6.55}$$

and

$$- \int_\Omega \rho_D(t) : \varphi H^p_{n_k,sym}(t) \, dx = \langle \mathcal{L}(t), \varphi w(t) \rangle - \int_\Omega \rho(t) : \varphi E w(t) \, dx$$

$$
\begin{aligned}
&- \int_\Omega \rho(t) : (\nabla\varphi \odot w(t)) \; dx + \int_\Omega \rho(t) : \varphi H^e_{n_k,sym}(t) \; dx \\
&+ \int_\Omega \rho(t) : (\nabla\varphi \odot u_{n_k}(t)) \; dx - \langle \mathcal{L}(t), \varphi u_{n_k}(t) \rangle \, . \qquad (6.56)
\end{aligned}
$$

It is crucial to observe that φu_{n_k} is zero on Γ_N, thus, for every $\tau \in [0, t]$

$$
\begin{aligned}
\langle \dot{\mathcal{L}}(\tau), \varphi u_{n_k}(\tau) \rangle &\to \langle \dot{\mathcal{L}}(\tau), \varphi u(\tau) \rangle \, , \\
\langle \mathcal{L}(\tau), \varphi u_{n_k}(\tau) \rangle &\to \langle \mathcal{L}(\tau), \varphi u(\tau) \rangle \, .
\end{aligned}
$$

By (6.53)-(6.56), the convergences (6.35)-(6.36), and Proposition 2.2 in [15] we deduce

$$
\begin{aligned}
\liminf_{k\to\infty} \left[- \int_0^t \int_\Omega \rho_D(\tau) : \varphi \dot{H}^p_{n_k,sym}(\tau) \; dx d\tau \right] &= \int_{\Omega\cup\Gamma_D} \varphi \; d[\rho_D(0) : H^p(0)] \\
- \int_{\Omega\cup\Gamma_D} \varphi \; d[\rho_D(t) : H^p(t)] &+ \int_0^t \int_{\Omega\cup\Gamma_D} \varphi \; d[\dot{\rho}_D(\tau) : H^p(\tau)] \; d\tau \, . \qquad (6.57)
\end{aligned}
$$

Combining (6.52) and (6.57) and passing to the limit as $\varphi \to \mathbb{1}_{\Omega\cup\Gamma_D}$ we finally obtain (6.50), concluding the proof. ■

Applying Theorem 4.7 in [15] the opposite energy inequality can be proved, showing that the map

$$
t \mapsto (u(t), H^e(t), H^p(t))
$$

is an energetic solution to the PR model.

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